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Pablo dos Santos Corrêa Junior

Homoclinic Solution to Zero of a Non-autonomous, Nonlinear, Second Order Differential Equation with Quadratic Growth on the Derivative

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Dissertation presented to the Program of Academic Master's Degree in Mathematics from Federal University of Juiz de Fora as required to obtain Master's Degree in Mathematics.

Advisor: Dr. Luiz Fernando de Oliveira Faria

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Vi por mandado da Santa e Geral Inquisição [...] e não achei neles coisa alguma escandalosa, nem contrária à **f**é e bons constumes, somente me pareceu que era necessário advertir os leitores que o Autor [...] usa de uma ficção dos deuses d**o**s gentios. E ainda que Santo Agostinho nas suas Retratações se **r**etrate de ter chamado, nos livros que compôs, De Ordine, às Musas deusas, todavia como isto é Poesi**a** e fingimento, e o Autor, como poeta, não pretenda mais que ornar o estilo poético, não tivemos por inconveniente ir esta fá**b**ula dos deuses na obra, c**o**nhecendo-a por ta**l**. E ficando sempre **s**alva a verdade de n**o**ssa sa**n**t**a** fé, que todos os deuses dos gentios são demônios. E por isso me pa**r**eceu o livro digno de se imprimir [...]. Em fé d**o** qual assinei aqui.

Frei Bartolomeu Ferreira

(Os Lusíadas)

ABSTRACT

The aim of this work is to obtain a positive, smooth, even and homoclinic solution to the problem

$$-(A(u)u')'(t) + u(t) = \lambda a_1(t)|u(t)|^{q-1} + |u(t)|^{p-1} + g(|u'(t)|), \quad \text{in } \mathbb{R}.$$

Considering $1 < q < 2 < p < +\infty$ and $a_1 \in L^s(\mathbb{R}) \cap C(\mathbb{R})$, $\mathfrak{s} = \frac{2}{2-q}$, a positive even function.

Also $A : \mathbb{R} \to \mathbb{R}$ a Lipschitz, smooth (at least $C^1(\mathbb{R})$), nondecreasing function satisfying

$$\exists \gamma \in (0,1) \text{ such that } 0 < \gamma \leq A(t) \quad \forall t \in \mathbb{R},$$

and $g : \mathbb{R} \longrightarrow \mathbb{R}$ a continuous function satisfying

$$0 \leq sg(s) \leq |s|^\theta \text{ for all } s \in \mathbb{R}, \text{ where } 2 < \theta \leq 3.$$

By homoclinic we mean "homoclinic to the origin" or "homoclinic to zero" , i.e, the solution must verify $\lim_{x\to\pm\infty} u(x) = 0$.

Keywords: Galerkin Method. Homoclinic Solution. Quadratic Growth on the Derivative. Differential Equation.

RESUMO

O objetivo principal deste trabalho é obter uma solução positiva, suave, par e homoclínica para o problema

$$-(A(u)u')'(t) + u(t) = \lambda a_1(t)|u(t)|^{q-1} + |u(t)|^{p-1} + g(|u'(t)|), \quad \text{em } \mathbb{R}.$$

Considerando $1 < q < 2 < p < +\infty$ e $a_1 \in L^{\mathfrak{s}}(\mathbb{R}) \cap C(\mathbb{R})$, $\mathfrak{s} = \frac{2}{2-q}$, uma função positiva e par.

Também $A : \mathbb{R} \to \mathbb{R}$ uma função Lipschitz, suave (minímo $C^1(\mathbb{R})$), não decrescente e satisfazendo

$$\exists \gamma \in (0,1) \text{ tal que } 0 < \gamma \leq A(t) \quad \forall t \in \mathbb{R},$$

e $g : \mathbb{R} \to \mathbb{R}$ uma função contínua satisfazendo

$$0 \leq sg(s) \leq |s|^{\theta} \text{ para todo } s \in \mathbb{R}, \text{ onde } 2 < \theta \leq 3.$$

Por homoclínica estamos nos referindo a "homoclínica para a origem" ou "homoclínica para zero", isto é, a solução deve verificar $\lim_{x \to \pm\infty} u(x) = 0$.

Palavras-chave: Método de Galerkin. Solução Homoclínica. Crescimento Quadrático na Derivada. Equação Diferencial.

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1 INTRODUCTION

The aim of this work is to obtain a positive, smooth, even and homoclinic solution to the problem

$$-(A(u)u')'(t) + u(t) = \lambda a_1(t)|u(t)|^{q-1} + |u(t)|^{p-1} + g(|u'(t)|), \quad \text{in } \mathbb{R}. \tag{1.1}$$

Considering $1 < q < 2 < p < +\infty$ and $a_1 \in L^{\mathfrak{s}}(\mathbb{R}) \cap C(\mathbb{R})$, $\mathfrak{s} = \frac{2}{2-q}$, a positive even function.

Also $A : \mathbb{R} \to \mathbb{R}$ a Lipschitz, smooth (at least $C^1(\mathbb{R})$), nondecreasing function satisfying

$$\exists \gamma \in (0,1) \text{ such that } 0 < \gamma \leq A(t) \quad \forall t \in \mathbb{R}, \tag{1.2}$$

and $g : \mathbb{R} \longrightarrow \mathbb{R}$ a continuous function satisfying

$$0 \leq sg(s) \leq |s|^{\theta} \text{ for all}\, s \in \mathbb{R}, \text{ where } 2 < \theta \leq 3. \tag{1.3}$$

By homoclinic we mean “homoclinic to the origin” or “homoclinic to zero” , i.e, the solution must verify $\lim_{x\to\pm\infty} u(x) = 0$.

1.1 MOTIVATION FOR THE PROBLEM

The idea to consider this problem came after a careful reading of the article [1], where the authors considered a similar equation but with a different set of hypothesis; namely their formulation was focused in the study of the equation

$$\begin{cases} -(A(u)u')' + u(t) = h(t, u(t)) + g(t, u'(t)) \text{ in } \mathbb{R} \\ u(\pm\infty) = u'(\pm\infty) = 0, \end{cases}$$

with

(H_1) $h, g : \mathbb{R}^2 \to \mathbb{R}$ locally Hölder continuous, even on the first variable and $h(t,0) = g(t,0) = 0$;

(H_2) there exist constants $0 < r_1, r_2 < 1$ and smooth functions $b \in L^1(\mathbb{R}) \cap L^{\infty}(\mathbb{R})$ with $b(t) > 0$ for all $t \in \mathbb{R}$, $a_1 \in L^2(\mathbb{R})$ and $a_2 \in L^{\frac{2}{1-r_2}}(\mathbb{R})$, satisfying

$$b(t)|\mu|^{r_1} \leq h(t,\mu) \leq a_1(t) + a_2(t)|\mu|^{r_2}, \quad \forall (t,\mu) \in \mathbb{R}^2;$$

(H_3) there exist a constant $0 < r_3 < 1$ and smooth functions $a_3 \in L^{\frac{2}{1-r_3}}(\mathbb{R})$ and $a_4 \in L^2(\mathbb{R})$ satisfying

$$0 \leq g(t,\eta) \leq a_4(t) + a_3(t)|\eta|^{r_3} \quad \forall (t,\eta) \in \mathbb{R}^2;$$

(H_4) the function A is smooth, nondecreasing and there exists $\gamma \in (0,1)$ satisfying

$$0 < \gamma \leq A(t) \quad \forall t \in \mathbb{R}.$$

Comparing with our work, we considered sup-linear growth on u and u', terms involving this type of growth are not covered in [1]. Another aspect that we would like to emphasize is the weakening of the hypothesis over g: comparing with [1], we asked only for continuity over g, instead of Hölder continuity.

Although the formulation presented here is not an immediate consequence of [1], some techniques therein proved to be quite solid and very useful in the study of this type of problem, transcending the circumstances framed by the authors. In this respect, we would like to emphasize the [1, Thm. 3.1], which became (in this dissertation) Theorem 19 and the final part of the arguments presented in [1, Thm. 4.3]; these arguments were used here to prove Proposition 31.

1.2 MAIN HURDLES

Our formulation presented some interesting challenges, for instance, the problem is not variational. Among the non-variational techniques we chose the Galerkin Method as a tool to gather information about existence of weak solutions. Although proving itself beneficial, the Galerkin Method presented us with other types of challenges to circumvent. For example, the nonlinear term $g(|u'(t)|)$ with $0 \leq sg(s) \leq |s|^\theta$ and $2 < \theta \leq 3$ enables us to take $g(s) \equiv sign(s)|s|^2$. Thus estimations involving $\int_\Omega |u'|^2$ become essential to the calculations but, at the same time, a mystery: this is due to the lack of information about u', since all embedding theorems of $H_0^1(\Omega)$ do not provide a substantial information about u' as it does for u.

We consider the case $\theta = 3$ as the *critical* one and treat it separately in our estimations. For $\theta > 3$ we would get expressions involving $\int |u'|^{\theta-1}$ that we could not control, because $\theta - 1 > 2$ and we only know that $u' \in L^2$; for this reason, we limited $\theta \leq 3$, and $\theta > 2$ was required because we wanted to focus on the sup-linear case.

There are some literature about equations on domains in $\mathbb{R}^n$ involving the term $|\nabla u|^2$ in the nonlinearity (see [2, 3, 4]), some authors call this type of growth : "*critical growth on the gradient*". Simple changes in how this term appears in the equation can have dramatic effects on the outcome. For instance, a simple change in the sign of $|\nabla u|^2$ can lead to a total failure to obtain a solution (even in the weak sense), see the article [3] for more information. With that been said, we consider the possibility to take $g(s) \equiv sign(s)|s|^2$ a major contribution of our work to the study of equations following the same type as (1.1).

The methods applied in our work require certain symmetry, which is due mainly to a lack of a comparison principle (known to the author) to guarantee that some limit-functions are not zero almost everywhere (a.e). This necessity is expressed in the hypothesis of Theorem 19; with emphasis on the items *2* and *4*. To overcome this obstacle we founded this work focusing on the set $\mathbb{E}_0^1(I) = \{u \in H_0^1(I); u(t) = u(-t) \text{ a.e } \}$, $I \subset \mathbb{R}$ an interval, which is the subset of $H_0^1(I)$ consisting of even functions. This set can be understood as the set of radial symmetric functions in $\mathbb{R}$.

1.3 STRAUSS APPROXIMATION

In order to develop our study, we used the Galerkin Method to construct a solution to a problem approximating (1.1) in a bounded interval; this approximation is due to a sequence of Lipschitz functions (f_k) converging to g that has been called *Strauss Approximation*, after appearing in the article [5]. This approximation was useful because it helped us to work with the necessary estimations without extra hypothesis over g. In detail, our approximation was

$$\begin{cases} -(A(u)u')'(t) + u(t) = \lambda a_1(t)|u(t)|^{q-1} + |u(t)|^{p-1} + f_k(|u'(t)|) + \frac{\psi}{k}, & \text{in } (-n, n) \\ u(n) = u(-n) = 0, \end{cases} \tag{1.4}$$

considering $\psi \in L^2(-n, n)$ a positive and even function; the term $\frac{\psi}{k}$ was utilized to guarantee that a solution of (1.4) is not identically null. Regarding this approximation, we proved that

Theorem. There exist $\lambda^* > 0$, $\beta \in (0, 1)$ and $k^* \in \mathbb{N}$ for which the problem (1.4) admits a nontrivial, even, non-negative $C^{1,\beta}[-n, n] \cap C^2(-n, n)$ solution for every $\lambda \in (0, \lambda^*)$ and $k \geq k^*$.

The proof of this theorem is a direct consequence of Proposition 27 and Proposition 29.

We followed [6] in the definition and presentation of the properties of the sequence (f_k). In this article, the authors used this approximation to avoid the usage of Ambrosetti-Rabinowitz condition and were able to obtain a positive solution to the equation

$$\begin{cases} -\Delta u = \lambda u^{q(r)-1} + f(r, u) & \text{in } B(0, 1) \\ u > 0 & \text{in } B(0, 1) \\ u = 0 & \text{on } \partial B, \end{cases}$$

see [6] for more information.

We would like to emphasize that, in [6], the authors used this approximation in a term involving u; namely they used it to approximate $f(r, u)$. In our work we used it in u'. As far as we can tell, this is the first time that this approximation was used in a term involving the *derivative* of u.

After this step we were able to prove

Theorem. The equation

$$\begin{cases} -(A(u)u')'(t) + u(t) = \lambda a_1(t)|u(t)|^{q-1} + |u(t)|^{p-1} + g(|u'(t)|), & \text{in } (-n, n) \\ u(n) = u(-n) = 0, \end{cases} \tag{1.5}$$

has, for all $\lambda \in (0, \lambda^*)$, a positive $C^{1,\alpha}[-n, n] \cap C^2(-n, n)$ even solution for some $\alpha \in (0, \beta)$.

This was proved in Proposition 31 .

With this theorem we obtained a sequence (u_n) of solutions in $W^{1,2}(\mathbb{R})$, (indexed with “n” from the interval $(-n, n)$), that had converging subsequences when restricted to intervals such as $K_j = [-j, j]$, and these candidates (the limits of the subsequences) could be extended to a solution in $\mathbb{R}$ of (1.1). Using [7, Thm. 1] and some standard arguments, we were able to prove that

Theorem. The equation

$$-(A(u)u')'(t) + u(t) = \lambda a_1(t)|u(t)|^{q-1} + |u(t)|^{p-1} + g(|u'(t)|), \quad \text{in } \mathbb{R}. \tag{1.6}$$

has an even, positive and $C^2(\mathbb{R})$ solution that satisfies

$$\lim_{x \to \pm\infty} u(x) = 0$$

for all $\lambda \in (0, \lambda^*)$.

The construction of such solution was the main goal of Section 4.3.

1.4 FURTHER READING

In literature one can find a range of equations similar to (1.1) also seeking homoclinic or heteroclinic solutions. We would like to, briefly, expose some of these works:

- Article [8] studies the existence of homoclinic solution, using arguments of lower and upper solutions and fixed point theorem, for the equation

  $$u''(t) - ku(t) = f(t, u(t), u'(t)) \text{ a.e.t } \in \mathbb{R} \tag{1.7}$$

  considering $f : \mathbb{R}^3 \to \mathbb{R}$ a L^1-Carathédory function, that is, f verifies:

  (i) for each $(x, y) \in \mathbb{R}^2$, $t \mapsto f(t, x, y)$ is measurable on $\mathbb{R}$;

  (ii) for almost every $t \in \mathbb{R}$, $(x, y) \mapsto f(t, x, y)$ is continuous on $\mathbb{R}^2$;

  (iii) for each $\rho > 0$, there exists a positive function $\varphi_\rho \in L^1(\mathbb{R})$ such that, whenever $x, y \in [-\rho, \rho]$, then

  $$|f(t, x, y)| \leq \varphi_\rho(t), \text{ a.e } t \in \mathbb{R}.$$

  Comparing with our work we have proposed new arguments and utilized the weight $A(u)$ on the therm involving the second order derivative, not covered in [8]. This article also presents some examples and applications of the Duffing-type equation, related to this topic.

- Article [9] investigates the existence of positive solution to the equation

  $$u'' + cu'a(t)f(u) = 0$$

  in unbounded intervals. The author considered $c > 0$ and $f(x), a(t)$ non-negative functions; moreover, the assumptions over these functions were:

(H1) $f : \mathbb{R}_+ \to \mathbb{R}_+$ is a locally Lipschitz function such that $f(0) = 0$, and there exists $d > 0$ such that $f(x) > 0$ if $0 < x < d$.

(H2) $a : \mathbb{R} \to \mathbb{R}$ is a continuous function such that, for some $\alpha > 0$; $\alpha < a(t) \leq 1$.

The author analyzed two types of problems of his interest:

Type 1 f has a zero, e.g. $f(1) = 0$, with $f(u) > 0$ whenever $0 < u < 1$. In this case, he searched for heteroclinic solutions which where strictly decreasing.

Type 2 Whether the proposed problem has a non-trivial positive solution defined in an unbounded interval such as $[t_0, +\infty]$ and satisfying $u(t_0) = 0 = u(+\infty)$.

- Article [1] was previously discussed but, for completeness, we also mention it here.
- Article [10] searches heteroclinic solution for the problem

$$(a(x(t))x'(t))' = f(t, x(t), x'(t)) \text{ a.e } t \in \mathbb{R} \tag{1.8}$$

considering $a(x)$ a non-linear, continuous and positive function and $f : \mathbb{R}^3 \to \mathbb{R}$ a nonlinear Carathédory function. In some aspect we think that is article, the article [1] and our work complement each other in the investigation of equations of type (1.8).

1.5 DISPOSITION OF THE CHAPTERS

Chapter 2 deals with some basic properties of Sobolev Spaces; the main result therein is an embedding theorem used afterwards.

Chapter 3 deals with the basic definition of Hölder Spaces and, again, the goal here is to prove an embedding theorem.

Chapter 4 is where we treat the stated problem. We recommend the reader some basic knowledge in *measure theory and functional analysis*; objectively, the topics covered from chapters 1 to 8 of [11].

2 SOBOLEV'S SPACES

Our main goal in this chapter is to prove the following theorem:

Theorem. There exists a constant C (depending only on $|I| \leq \infty$) such that

1. $\|u\|_{L^\infty(I)} \leq C\|u\|_{W^{1,p}(I)} \forall u \in W^{1,p}(I)$, $\forall 1 \leq p \leq \infty$.

In other words, $W^{1,p}(I) \hookrightarrow L^\infty(I)$ with continuous injection for all $1 \leq p \leq \infty$.

Further, if I is *bounded* then

2. the injection $W^{1,p}(I) \hookrightarrow C(\overline{I})$ is *compact* for all $1 < p \leq \infty$,
3. the injection $W^{1,1}(I) \hookrightarrow L^q(I)$ is compact for all $1 \leq q < \infty$.

This theorem was used multiple times throughout the argumentation of chapter 4. We will follow the order and arguments from chapter 8 of [11]. The goal of this chapter is to only discuss some basic properties of the Soblev Spaces in the context of $\mathbb{R}$ so, to a thorough explanation, consider reading this great book.

2.1 DEFINITION AND BASIC PROPERTIES

Given $I \subset \mathbb{R}$ an open interval and $1 \leq p \leq \infty$ we define

$$W^{1,p}(I) = \{u \in L^p(I); \exists g \in L^p(I) \text{ such that } \int_I u\varphi' dt = -\int_I g\varphi dt, \quad \forall \varphi \in C_0^\infty(I)\}. \tag{2.1}$$

These classes of sets are called Sobolev spaces. If $p = 2$ we may use the notation $H^1(I) = W^{1,2}(I)$. These sets are normed vector spaces with the norms – which are all equivalents– given by

1. $\|u\|_{W^{1,p}} = \|u\|_{L^p} + \|u'\|_{L^p}$;
2. for $1 < p < \infty$, $\|u\|_{W^{1,p}} = \left(\|u\|_{L^p}^p + \|u'\|_{L^p}^p\right)^{1/p}$.

   Moreover, we have that:

   - the space $W^{1,p}$ is a Banach space for $1 \leq p \leq \infty$;
   - it is reflexive for $1 < p < \infty$;
   - it is separable for $1 \leq p < \infty$;
   - H^1 is a separable Hilbert space, with inner product given by :

$$\langle u, v\rangle_{H^1} = \langle u, v\rangle_{L^2} + \langle u', v'\rangle_{L^2}.$$

Lemma 1. *Let $f \in L^1_{loc}(I)$ be such that*

$$\int_I f\varphi' dt = 0 \quad \forall \varphi \in C_0^\infty(I).$$

Then there exists a constant C such that $f = C$ a.e *on I.*

Proof. See [11, Pag. 204, Lemma 8.1]. □

Lemma 2. *Let $g \in L^1_{loc}(I)$; for $y_0 \in I$ fixed, set*

$$v(x) = \int_{y_0}^{x} g(t)dt, \quad x \in I.$$

Then $v \in C(I)$ and

$$\int_I v\varphi' dt = -\int_I g\varphi dt \quad \forall \varphi \in C_0^\infty(I).$$

Proof. See [11, Pag. 205, Lemma 8.2] □

Theorem 3. *Let $u \in W^{1,p}(I)$ with $1 \leq p \leq \infty$, and I bounded or unbounded; then there exists a function $\tilde{u} \in C(\overline{I})$ such that $u = \tilde{u}$ a.e on I, and*

$$\tilde{u}(x) - \tilde{u}(y) = \int_y^x u'(t)dt \quad \forall x, y \in \overline{I}.$$

Proof. Let $y_0 \in I$ and define $\overline{u}(x) = \int_{y_0}^x u'(t)dt$. By Lemma 2 we have

$$\int_I \overline{u}\varphi' dt = -\int_I u'\varphi dt \quad \forall \varphi \in C_0^\infty(I).$$

Thus, $\int_I (u - \overline{u})\varphi' = 0$ for all $\varphi \in C_0^\infty(I)$. It follows from Lemma 1 that there exist a constant C such that $u - \overline{u} = C$ *a.e* on I. Then, taking $\tilde{u}(x) = \overline{u}(x) + C$ we finish the proof. □

Proposition 4. *Let $u \in L^p(I)$ with $1 < p \leq \infty$. The following properties are equivalent:*

(i) $u \in W^{1,p}(I)$;

(ii) *there is a constant C such that*

$$\left| \int_I u\varphi' dt \right| \leq C\|\varphi\|_{L^{p'}(I)} \quad \forall \varphi \in C_0^\infty(I)$$

where $1/p + 1/p' = 1$.

Furthermore, we can take $C = \|u'\|_{L^p(I)}$.

Proof. $(i) \Rightarrow (ii)$ follows from the definition of Sobolev Spaces.

$(ii) \Rightarrow (i)$ Define the linear functional

$$\varphi \in C_0^\infty(I) \mapsto \int_I u\varphi' dt$$

and notice that, since $p' < \infty$, $C_0^\infty(I)$ is dense in $L^{p'}(I)$ [See 11, pag. 97, thm. 4.12]. This functional is continuous for the $L^{p'}(I)$ norm and, therefore, by Hahn-Banach theorem there exists an extension F defined on $L^{p'}(I)$. By the Riesz representation theorem there exist $g \in L^p(I)$ such that

$$F(\varphi) = \int_I g\varphi dt \quad \forall \varphi \in L^{p'}(I).$$

In particular,

$$\int_I u\varphi' dt = \int_I g\varphi dt \quad \forall \varphi \in C_0^\infty(I),$$

thus $u \in W^{1,p}(I)$. □

Proposition 5. *A function $u \in L^\infty(I)$ belongs to $W^{1,\infty}(I)$ if and only if there exists a constant C such that*

$$|u(x) - u(y)| \leq C|x - y| \text{ for a.e. } x, y \in I.$$

Proof. If $u \in W^{1,\infty}(I)$ we can apply Theorem 3 to obtain:

$$|u(x) - u(y)| \leq \|u'\|_{L^\infty}|x - y| \text{ for a.e. } x, y \in I.$$

Conversely, let $\varphi \in C_0^\infty(I)$. Since φ has compact support, for $|h|$ small enough the following integral is well defined:

$$\int_I [u(x+h) - u(x)]\varphi(x)dx.$$

Moreover, we have that

$$\int_I [u(x+h) - u(x)]\varphi(x)dx = \int_I u(x)[\varphi(x-h) - \varphi(x)]dx.$$

Thus,

$$\left|\int_I u(x)[\varphi(x-h) - \varphi(x)]dx\right| = \left|\int_I [u(x+h) - u(x)]\varphi(x)dx\right| \leq C|h|\|\varphi\|_{L^1}.$$

Dividing by $|h|$ and letting $h \to 0$ we obtain

$$\left|\int_I u\varphi' dt\right| \leq C\|\varphi\|_{L^1} \quad \forall \varphi \in C_0^\infty(I).$$

Thus, by Proposition 4 we conclude that $u \in W^{1,\infty}(I)$. □

Definition 6. Let $v : \mathbb{R} \to \mathbb{R}$ and $h \in \mathbb{R}$. We define the function $\tau_h v$ by $(\tau_h v)(x) = v(x+h)$.

Proposition 7. *Let $u \in L^p(\mathbb{R})$ with $1 < p < \infty$. The following properties are equivalent:*

(i) $u \in W^{1,p}(\mathbb{R})$;

(ii) there exists a constant C such that, for all $h \in \mathbb{R}$,

$$\|\tau_h u - u\|_{L^p(\mathbb{R})} \leq C|h|.$$

Moreover, one can choose $C = \|u'\|_{L^p(\mathbb{R})}$.

Proof. $(i) \Rightarrow (ii)$ By Theorem 3, for all $x, h \in \mathbb{R}$ we have:

$$u(x+h) - u(x) = \int_x^{x+h} u'(t)dt = h\int_0^1 u'(x+sh)ds.$$

Thus, by Hölder's inequality we have

$$|u(x+h) - u(x)|^p \leq |h|^p \int_0^1 |u'(x+sh)|^p ds.$$

Then,

$$\begin{aligned}\int_{\mathbb{R}} |u(x+h)-u(x)|^p dx &\leq |h|^p \int_{\mathbb{R}} \int_0^1 |u'(x+sh)|^p ds dx \\ &= |h|^p \int_0^1 \int_{\mathbb{R}} |u'(x+sh)|^p dx ds \\ &= |h|^p \int_0^1 \|u'\|_{L^p}^p ds = |h|^p \|u'\|_{L^p(I)}^p.\end{aligned}$$

and so we obtain the inequality (ii).

$(ii) \Rightarrow (i)$ Let $\varphi \in C_0^\infty(I)$. For all $h \in \mathbb{R}$ we have

$$\int_{\mathbb{R}} [u(x+h)-u(x)]\varphi(x)dx = \int_{\mathbb{R}} u(x)[\varphi(x-h)-\varphi(x)]dx.$$

Using this equality, Hölder's inequality and (ii) we conclude

$$\left|\int_{\mathbb{R}} u(x)[\varphi(x-h)-\varphi(x)]dx\right| \leq C|h|\|\varphi\|_{L^{p'}(\mathbb{R})}.$$

Dividing by $|h|$ and letting $h \to 0$ we obtain

$$\left|\int_{\mathbb{R}} u\varphi' dt\right| \leq C\|\varphi\|_{L^{p'}(\mathbb{R})}.$$

By Proposition 4 we have that $u \in W^{1,p}(\mathbb{R})$. □

Theorem 8 (Extension Theorem)**.** *Let $1 \leq p \leq \infty$. There exists a bounded linear operator $P : W^{1,p}(I) \to W^{1,p}(\mathbb{R})$, called an extension operator, satisfying the following properties:*

(i) $Pu|_I = u \quad \forall u \in W^{1,p}(I)$;

(ii) $\|Pu\|_{L^p(\mathbb{R})} \leq C\|u\|_{L^p(I)} \forall u \in W^{1,p}(I)$;

(iii) $\|Pu\|_{W^{1,p}(\mathbb{R})} \leq C\|u\|_{W^{1,p}(I)} \quad \forall u \in W^{1,p}(I)$.

where C depends only on $|I| \leq \infty$.

Proof. See [11, Pag. 209,Thm. 8.6]. □

Theorem 9. *Let $u \in W^{1,p}(I)$ with $1 \leq p < \infty$. Then there exists a sequence (u_n) in $C_0^\infty(\mathbb{R})$ such that $u_n|_I \to u$ in $W^{1,p}(I)$.*

Proof. See [11, Pag. 211,Thm. 8.7]. □

Theorem 10. *There exists a constant C (depending only on $|I| \leq \infty$) such that*

1. $\|u\|_{L^\infty(I)} \leq C\|u\|_{W^{1,p}(I)} \forall u \in W^{1,p}(I)$, $\forall 1 \leq p \leq \infty$.

In other words, $W^{1,p}(I) \hookrightarrow L^\infty(I)$ with continuous injection for all $1 \leq p \leq \infty$.

Further, if I is bounded *then*

2. *the injection* $W^{1,p}(I) \hookrightarrow C(\overline{I})$ *is* compact *for all* $1 < p \leq \infty$*,*

3. *the injection* $W^{1,1}(I) \hookrightarrow L^q(I)$ *is compact for all* $1 \leq q < \infty$*.*

Remark 1. To prove this theorem we will need the following result:

Theorem. Let $\mathcal{F}$ be a bounded set in $L^p(\mathbb{R}^n)$, with $1 \leq p < \infty$. Assume that

$$\lim_{|h|\to 0} \|\tau_h f - f\|_{L^p} = 0 \text{ uniformly in } f \in \mathcal{F},$$

i.e, for all $\epsilon > 0$ there exists $\delta > 0$ such that $\|\tau_h f - f\|_{L^p} < \epsilon$ for all $f \in \mathcal{F}$, for all $h \in \mathbb{R}^n$ with $|h| < \delta$.

Then the closure of $\mathcal{F}|_\Omega$ in $L^p(\Omega)$ is compact for any measurable set $\Omega \subset \mathbb{R}^n$ with finite measure. ($\mathcal{F}|_\Omega$ denotes the restriction to Ω of the functions in $\mathcal{F}$).

The proof of this theorem can be found in [11, pag. 111, thm. 4.26].

Proof. We'll begin proving item *1* for $I = \mathbb{R}$; the general case follows from this case and the extension theorem. Let $v \in C_0^\infty(\mathbb{R})$; if $1 \leq p < \infty$ set $G(s) = |s|^{p-1}s$. The function $G(v) \in C^\infty(\mathbb{R})$ and

$$(G(v))' = G'(v)v' = p|v|^{p-1}v'.$$

Then, for all $x \in \mathbb{R}$, we have

$$|v(x)|^{p-1}v(x) = G(v(x)) = \int_{-\infty}^{x} p|v(t)|^{p-1}v'(t)dt,$$

thus, using Hölder's inequality

$$\begin{aligned} |v(x)|^p &\leq p\int_{-\infty}^{x} |v(t)|^{p-1}|v'(t)|dt \\ &\leq p\left(\int_{\mathbb{R}} |v'(t)|^p dt\right)^{1/p} \cdot \left(\int_{\mathbb{R}} |v(t)|^{(p-1)p'} dt\right)^{1/p'} \\ &= p\|v'\|_{L^p(\mathbb{R})}\|v\|_{L^p(\mathbb{R})}^{p-1} \\ &\leq p\|v\|_{W^{1,p}}^p. \end{aligned}$$

and so we can obtain

$$\|v\|_{L^\infty(\mathbb{R})} \leq C\|v\|_{W^{1,p}}, \tag{2.2}$$

where C is a universal constant independent of p, because $p^{1/p} \leq e^{1/e}$ for all $p \geq 1$.

Now, given $u \in W^{1,p}(\mathbb{R})$, there exists a sequence (u_n) in $C_0^\infty(\mathbb{R})$ such that $u_n \to u$ in $W^{1,p}(\mathbb{R})$. Using (2.2) we see that (u_n) is a Cauchy sequence in $L^\infty(\mathbb{R})$; thus $u_n \to u$ in $L^\infty(\mathbb{R})$ and we obtain the item *1*.

Proof of *2*. Let $\mathbf{B}$ be the unit ball in $W^{1,p}(I)$, with $1 < p \leq \infty$. For $u \in \mathbf{B}$ we have

$$|u(x) - u(y)| = \left|\int_x^y u'(t)dt\right| \leq \|u'\|_{L^p(I)}|x-y|^{1/p'} \leq |x-y|^{1/p'} \quad \forall x, y \in I. \tag{2.3}$$

Then, from Arzelà-Ascoli theorem, $\mathbf{B}$ has a compact closure in $C(\overline{I})$.

Proof of *3*. Let $\mathbf{B}$ be the unit ball in $W^{1,1}(I)$ and P be the extension operator. Define $\mathbf{D} = P(\mathbf{B})$, so that $\mathbf{B} = \mathbf{D}|_I :=$ restriction to I of the functions in $\mathbf{D}$. By the Theorem 8 $\mathbf{D}$ is bounded in $W^{1,1}(\mathbb{R})$; therefore $\mathbf{D}$ is also bounded in $L^q(\mathbb{R})$, due to the interpolation inequality, since it is bounded both in $L^1(\mathbb{R})$ and in $L^\infty(\mathbb{R})$.

By Proposition 7, for every $f \in \mathbf{D}$ we have

$$\|\tau_h f - f\|_{L^1(\mathbb{R})} \leq |h|\|f'\|_{L^1(\mathbb{R})} \leq C|h|,$$

since $\mathbf{D}$ is a bounded subset of $W^{1,1}(\mathbb{R})$. Thus,

$$\begin{aligned}
\int_{\mathbb{R}} |\tau_h f - f|^q dt &= \int_{\mathbb{R}} |\tau_h f - f|^{q-1}|\tau_h f - f| dt && (2.4)\\
&\leq \int_{\mathbb{R}} (|\tau_h f| + |f|)^{q-1}|\tau_h f - f| dt && (2.5)\\
&\leq (2\|f\|_{L^\infty(\mathbb{R})})^{q-1} \int_{\mathbb{R}} |\tau_h f - f| dt && (2.6)\\
&= (2\|f\|_{L^\infty(\mathbb{R})})^{q-1} \|\tau_h f - f\|_{L^1(\mathbb{R})} \leq C|h|. && (2.7)
\end{aligned}$$

Then

$$\|\tau_h f - f\|_{L^q(\mathbb{R})} \leq C|h|^{1/q}$$

and $\lim_{|h|\to 0} \|\tau_h f - f\|_{L^q(\mathbb{R})} = 0$. Consequently, by [11, pag. 111, thm. 4.26] we obtain the desired result. □

Remark 2. Let I be a bounded interval, then $W^{1,2}(I) \hookrightarrow L^2(I)$ is compact.

Indeed, let $\mathbf{B}$ be the unit ball in $W^{1,2}(I)$ and $P : W^{1,2}(I) \to W^{1,2}(\mathbb{R})$ the extension operator. Define $\mathbf{D} = P(\mathbf{B})$. By Theorem 8 $\mathbf{D}$ is bounded in $W^{1,2}(\mathbb{R})$ and this implies that $\mathbf{D}$ is bounded in $L^2(\mathbb{R})$. Using Proposition 7 we have, for all $f \in \mathbf{D}$

$$\|\tau_h f - f\|_{L^2(\mathbb{R})} \leq \|f'\|_{L^2(\mathbb{R})}|h| \leq C|h|,$$

where C is a constant independent of f (C is the constant of the extension operator). Thus $\lim_{|h|\to 0} \|\tau_h f - f\|_{L^2(\mathbb{R})} = 0$; by [11, pag. 111, thm. 4.26] we conclude that the embedding is compact.

Corollary 11. *Suppose that I is an unbounded interval and $u \in W^{1,p}(I)$ with $1 \leq p < \infty$. Then*

$$\lim_{|x|\to\infty} u(x) = 0.$$

Proof. See [11, Pag. 214, Corl. 8.9]. □

Definition 12. Given $1 \leq p < \infty$, denote by $W_0^{1,p}(I)$ the closure of $C_0^\infty(I)$ in $W^{1,p}(I)$. Set

$$H_0^1(I) = W_0^{1,2}(I).$$

The space $W_0^{1,p}(I)$ is equipped with the norm of $W^{1,p}(I)$, and the space $H_0^1(I)$ is equipped with the inner product of $H^1(I)$. The space $W_0^{1,p}(I)$ is a separable Banach space. Moreover, it is reflexive for $p > 1$; the space $H_0^1(I)$ is a separable Hilbert space.

Proposition 13 (Poincaré's inequality)**.** *Suppose I is a bounded interval. The there exists a constant C (depending on $|I| < \infty$) such that*

$$\|u\|_{W^{1,p}(I)} \leq C\|u'\|_{L^p(I)} \quad \forall u \in W_0^{1,p}(I).$$

In other words, on $W_0^{1,p}(I)$, the quantity $\|u'\|_{L^p(I)}$ is a norm equivalent to the $W^{1,p}(I)$ norm.

Proof. See [11, Pag. 218, Prop. 8.13]. □

3 HÖLDER'S SPACES

In order to obtain the solution of the proposed problem in chapter 4 we will need to juggle between multiple normed spaces called *Hölder spaces*; this will be useful because there are certain embedding theorems among them, providing a tool to obtain converging subsequences. For more information concerning Hölder Spaces see [12].

3.1 DEFINITION

Let Ω be an open subset of $\mathbb{R}^n$ and $f : \Omega \to \mathbb{R}$ a $k-$differentiable function in $x \in \Omega$. For $\gamma = (\gamma_1, \ldots, \gamma_n) \in \mathbb{N}^n$, we define $|\gamma| = \sum_{i=1}^n \gamma_i$ and

$$D^\gamma f(x) = \frac{\partial^{|\gamma|} f}{\partial x_n^{\gamma_n} \partial x_{n-1}^{\gamma_{n-1}} \cdots \partial x_1^{\gamma_1}}(x) \quad \forall |\gamma| \leq k.$$

Definition 14. (a) We say that f is a *Hölder continuous function with exponent* λ, or $\lambda-$Hölder continuous, in the set Ω when f satisfies

$$[f]_\lambda(\Omega) := \sup_{\substack{x,y\in\Omega \\ x\neq y}} \frac{|f(x) - f(y)|}{|x - y|^\lambda} < +\infty;$$

in this case, we say that $f \in C^\lambda(\Omega)$.

(b) We define:

$$C^k(\overline{\Omega}) = \{f \in C^k(\Omega); D^\gamma f \text{ is bounded and uniformly continuous in } \Omega \quad \forall |\gamma| \leq k\}$$

with $\|f\|_{C^k(\overline{\Omega})} = \max_{|\gamma|\leq k} \|D^\gamma f\|_{L^\infty(\Omega)}$; and

$$C^{k,\lambda}(\overline{\Omega}) = \{f \in C^k(\overline{\Omega}); D^\gamma f \in C^\lambda(\Omega) \quad \forall |\gamma| \leq k\}.$$

The sets $C^{k,\lambda}(\overline{\Omega})$ are vector spaces called *Hölder Spaces*; they are Banach with the norm

$$\|f\|_{C^{k,\lambda}(\overline{\Omega})} = \|f\|_{C^k(\overline{\Omega})} + \max_{|\gamma|\leq k} [D^\gamma f]_\lambda(\Omega).$$

Remark 3. We may use the notation $|\cdot|_{k+\lambda}$ for the norm $\|\cdot\|_{C^{k,\lambda}(\overline{\Omega})}$.

3.2 EMBEDDING THEOREM

Theorem 15 (Arzelà-Ascoli)**.** *Let K be a compact metric space and let $\mathcal{H}$ be a bounded subset of $C(K)$. Assume that $\mathcal{H}$ is uniformly equicontinuous, that is,*

$$\forall \epsilon > 0 \; \exists \delta > 0 \text{ such that } d(x_1, x_2) < \delta \Rightarrow |f(x_1) - f(x_2)| < \epsilon \quad \forall f \in \mathcal{H}.$$

Then the closure of $\mathcal{H}$ in $C(K)$ is compact.

Proof. See [13, Pag. 290, Thm. 47.1] □

Remark 4. The proof of the following theorem was extracted from [14, Pag. 175, Thm. 9.6]; currently this material has no English translation.

Theorem 16. *Let $\Omega \subset \mathbb{R}^n$ be an open set. Then, for all $k \in \mathbb{N}$ and $0 < \lambda < \beta \leq 1$ we have the following continuous embeddings:*

$$C^{k+1}(\overline{\Omega}) \hookrightarrow C^k(\overline{\Omega}) \tag{3.1}$$
$$C^{k,\lambda}(\overline{\Omega}) \hookrightarrow C^k(\overline{\Omega}) \tag{3.2}$$
$$C^{k,\beta}(\overline{\Omega}) \hookrightarrow C^{k,\lambda}(\overline{\Omega}). \tag{3.3}$$

If Ω is bounded, then the last two embeddings are compact; moreover if Ω is convex and bounded, then all embeddings are compact.

If Ω is convex, there are two more embeddings:

$$C^{k+1}(\overline{\Omega}) \hookrightarrow C^{k,1}(\overline{\Omega}) \tag{3.4}$$
$$C^{k+1}(\overline{\Omega}) \hookrightarrow C^{k,\lambda}(\overline{\Omega}), \tag{3.5}$$

the last one been compact if Ω is also bounded.

Proof. First we will show the existence of the continuous embeddings. From the clear inequalities

$$\|f\|_{C^k(\overline{\Omega})} \leq \|f\|_{C^{k+1}(\overline{\Omega})},$$
$$\|f\|_{C^k(\overline{\Omega})} \leq \|f\|_{C^{k,\lambda}(\overline{\Omega})},$$

we establish the embeddings (3.1) and (3.2).

Now notice that, for $x, y \in \Omega$, $0 < \lambda \leq \beta$ and $|\gamma| \leq k$:

If $0 < |x - y| \leq 1$ then $|x - y|^\lambda \geq |x - y|^\beta$. Thus

$$\sup_{\substack{x,y\in\Omega \\ 0<|x-y|\leq 1}} \frac{|D^\gamma f(x) - D^\gamma f(y)|}{|x - y|^\lambda} \leq \sup_{\substack{x,y\in\Omega \\ x\neq y}} \frac{|D^\gamma f(x) - D^\gamma f(y)|}{|x - y|^\beta} = [D^\gamma f]_\beta(\Omega).$$

If $|x - y| \geq 1$ then $|x - y|^\lambda \geq 1$. Thus

$$\sup_{\substack{x,y\in\Omega \\ |x-y|\geq 1}} \frac{|D^\gamma f(x) - D^\gamma f(y)|}{|x - y|^\lambda} \leq 2\|D^\gamma f\|_{C^0(\overline{\Omega})},$$

so we conclude that $\|f\|_{C^{k,\lambda}(\overline{\Omega})} \leq 3\|f\|_{C^{k,\beta}(\overline{\Omega})}$.

To obtain (3.4) and (3.5), suppose that Ω is convex. Let $f \in C^{k+1}(\overline{\Omega})$; given $x, y \in \Omega$ and $|\gamma| \leq k$, by the Mean Value Theorem there exists $z \in \Omega$ belonging to the segment line joining x and y such that

$$D^\gamma f(x) - D^\gamma f(y) = \nabla D^\gamma f(z)(x - y).$$

Then,

$$|D^\gamma f(x) - D^\gamma f(y)| \leq \|f\|_{C^{k+1}(\overline{\Omega})}|x-y|$$

for all $x, y \in \Omega$ and for all $|\gamma| \leq k$, implying that $f \in C^{k,1}(\overline{\Omega})$ and

$$\|f\|_{C^{k,1}(\overline{\Omega})} \leq \|f\|_{C^{k+1}(\overline{\Omega})}.$$

The embedding (3.5) follows from (3.3) and (3.4).

Now we will address the stated compactness. Assume that Ω is bounded; we start proving that the embedding (3.2) is compact for the case $k = 0$. Let (f_j) be a bounded sequence in $C^{0,\lambda}(\overline{\Omega}) = C^\lambda(\overline{\Omega})$, then there exist $M > 0$ such that $\|f_j\|_{C^\lambda(\overline{\Omega})} \leq M$ for all j. Then

$$|f_j(x)| \leq M \text{ for all } x \in \Omega \text{ and for all } j.$$

Thus (f_j) is uniformly bounded and

$$|f_j(x) - f_j(y)| \leq M|x-y|^\lambda \quad \forall x, y \in \Omega \text{ and } \forall j.$$

Therefore the sequence (f_j) is equicontinuous. By the Arzelà-Ascoli Theorem, (f_j) has a converging subsequence in $C^0(\overline{\Omega})$; this establish the compactness of the embedding (3.2) in the case $k = 0$. For $k > 0$, if (f_j) is a bounded sequence in $C^{k,\lambda}(\overline{\Omega})$, then it is a bounded sequence in $C^{0,\lambda}(\overline{\Omega})$. By the previous case ($k = 0$) (f_j) has a converging subsequence in $C^{0,\lambda}(\overline{\Omega})$ (that we still denote by (f_j)); thus there exists $f \in C^0(\overline{\Omega})$ such that $f_j \to f$ in $C^0(\overline{\Omega})$. Notice that, for $|\gamma| = 1$, $(D^\gamma f_j)$ is also bounded in $C^0(\overline{\Omega})$; then there exists a subsequence of (f_j) (that we still denote by (f_j)) and $f^1_\gamma \in C^0(\overline{\Omega})$, such that $D^\gamma f_j \to f^1_\gamma$ in $C^0(\overline{\Omega})$. Thus $f^1_\gamma = D^\gamma f$, because the convergences are uniform. Continuing this process of extracting subsequences, one concludes that, for all $|\gamma| \leq k$, $D^\gamma f_j \to D^\gamma f$ in $C^0(\overline{\Omega})$. Meaning that $f_j \to f$ in $C^k(\overline{\Omega})$, proving the compactness of the embedding (3.2).

To obtain the compactness of (3.3) we'll use the compactness of (3.2); notice that, if (f_j) is a bounded sequence in $C^{k,\beta}(\overline{\Omega})$, say $\|f_j\|_{C^{k,\beta}(\overline{\Omega})} \leq M$, then we have:

$$\begin{aligned}\frac{|D^\gamma f_j(x) - D^\gamma f_j(y)|}{|x-y|^\lambda} &= \left(\frac{|D^\gamma f_j(x) - D^\gamma f_j(y)|}{|x-y|^\beta}\right)^{\frac{\lambda}{\beta}} \cdot |D^\gamma f_j(x) - D^\gamma f_j(y)|^{1-\frac{\lambda}{\beta}} \\ &\leq M^{\frac{\lambda}{\beta}} |D^\gamma f_j(x) - D^\gamma f_j(y)|^{1-\frac{\lambda}{\beta}}.\end{aligned}$$

Thus,

$$[D^\gamma f_j]_\lambda(\Omega) \leq 2^{1-\frac{\lambda}{\beta}} M^{\frac{\lambda}{\beta}} \|D^\gamma f_j\|^{1-\frac{\lambda}{\beta}}_{C^0(\overline{\Omega})}.$$

Using (3.2) one obtains a converging subsequence of (f_j) in $C^k(\overline{\Omega})$. The above inequality implies that, each and every on of the partial derivatives of this subsequence converges in $C^{0,\lambda}(\overline{\Omega})$. Then this subsequence converges in $C^{k,\lambda}(\overline{\Omega})$. Finally, if Ω is convex and bounded, the compactness of (3.1) and (3.5) follows from the composition of the continuous embedding (3.4) with the compact embeddings (3.2) and (3.3) for the case $\lambda = 1$:

$$C^{k+1}(\overline{\Omega}) \hookrightarrow \underbrace{C^{k,1}(\overline{\Omega}) \hookrightarrow C^k(\overline{\Omega})}_{compact}$$

$$C^{k+1}(\overline{\Omega}) \hookrightarrow \underbrace{C^{k,1}(\overline{\Omega}) \hookrightarrow C^{k,\lambda}(\overline{\Omega})}_{compact}. \qquad \square$$

4 MAIN PROBLEM

This chapter deals with the main purpose of this work:

Theorem (Main problem)**.** Consider the equation

$$\begin{cases} -(A(u)u')'(t) + u(t) = \lambda a_1(t)|u(t)|^{q-1} + |u(t)|^{p-1} + g(|u'(t)|), & \text{in } \mathbb{R} \\ \lim_{x\to\pm\infty} u(x) = 0. \end{cases} \tag{4.1}$$

With:

(H_1) $1 < q < 2 < p < +\infty$ and $a_1 \in L^{\mathfrak{s}}(\mathbb{R}) \cap C(\mathbb{R})$, $\mathfrak{s} = \frac{2}{2-q}$ a positive even function;

(H_2) $A : \mathbb{R} \to \mathbb{R}$ a Lipschitz, smooth (at least $C^1(\mathbb{R})$), nondecreasing function satisfying:

$$\exists\, \gamma \in (0,1) \text{ such that } 0 < \gamma \le A(t) \quad \forall t \in \mathbb{R};$$

(H_3) $g : \mathbb{R} \to \mathbb{R}$ a continuous function satisfying:

$$0 \le sg(s) \le |s|^{\theta} \text{ for all } s \in \mathbb{R}, \text{ where } 2 < \theta \le 3. \tag{4.2}$$

Then there exist $\lambda^* > 0$ such that, for all $\lambda \in (0, \lambda^*)$, problem (4.1) has an even, positive and $C^2(\mathbb{R})$ solution.

To tackle the problem we reduced it to a simpler one: first obtain a solution to functions defined on an interval $(-n, n)$, with $n \in \mathbb{N}$. From there we were able to infer the existence of a solution to (4.1).

4.1 SOLUTION IN A BOUNDED INTERVAL

Consider the problem

$$\begin{cases} -(A(u)u')'(t) + u(t) = \lambda a_1(t)|u(t)|^{q-1} + |u(t)|^{p-1} + g(|u'(t)|), & \text{in } (-n,n) \\ u(n) = u(-n) = 0. \end{cases} \tag{P_n}$$

With the same set of hypothesis $(H_1), (H_2)$ and (H_3).

Remark 5. We will use the notation $\|\cdot\|_{W^{1,2}}$ for the usual norm of H_0^1 and for $(\|u\|_{L^2} + \|u'\|_{L^2})$ or $(\|u\|_{L^2} + \|u'\|_{L^2})^2$. Since these norms are equivalent the results will not change but the constants may. In most of the cases $\|u\|_{W^{1,2}} = \|u\|_{L^2} + \|u'\|_{L^2}$. We also emphasize that, when the context is clear, we will omit the domain in norms such as those from the spaces $L^p(-n, n)$.

Definition 17. We will call $w \in H_0^1(-n, n)$ a *weak solution* of (P_n) if

$$\int_{-n}^{n} A(w)w'v' + \int_{-n}^{n} wv = \int_{-n}^{n} \lambda a_1|w|^{q-1}v + \int_{-n}^{n} |w|^{p-1}v + \int_{-n}^{n} g(|w'|)v$$

for all $v \in H_0^1(-n, n)$.

Remark 6. The following construction of the sequence (f_k) is due to [5].

Define $G(s) = \int_0^s g(t)dt$ so that G is differentiable and $G'(s) = g(s)$. By means of G we shall construct a sequence of approximations of g by Lipschitz functions $f_k : \mathbb{R} \longrightarrow \mathbb{R}$. Let

$$f_k(s) = \begin{cases} -k[G(-k - \frac{1}{k}) - G(-k)], & \text{if } s \leq -k \\ -k[G(s - \frac{1}{k}) - G(s)], & \text{if } -k \leq s \leq \frac{-1}{k} \\ k^2 s[G(\frac{-2}{k}) - G(\frac{-1}{k})], & \text{if } \frac{-1}{k} \leq s \leq 0 \\ k^2 s[G(\frac{2}{k}) - G(\frac{1}{k})], & \text{if } 0 \leq s \leq \frac{1}{k} \\ k[G(s + \frac{1}{k}) - G(s)], & \text{if } \frac{1}{k} \leq s \leq k \\ k[G(k + \frac{1}{k}) - G(k)], & \text{if } s \geq k \end{cases} \tag{4.3}$$

Theorem 18 (Lemma 1 from [6]). *The sequence f_k as defined above satisfies:*

1. $sf_k(s) \geq 0$ *for all* $s \in \mathbb{R}$*;*
2. *for all* $k \in \mathbb{N}$ *there is a constant* $c(k)$ *such that* $|f_k(\xi) - f_k(\eta)| \leq c(k)|\xi - \eta|$*, for all* $\xi, \eta \in \mathbb{R}$*;*
3. f_k *converges uniformly to* g *in bounded sets.*

Remark 7. From the definition of the sequences f_k, and the fact that $sign(g(s)) = sign(s)\ \forall s \in \mathbb{R}$, it follows without difficulties that *1* is true. In [6, Pag. 6, Prop. 5] one can find a detailed proof of *2* , so we will only prove *3* by an alternative argumentation.

Proof. Let $m \in \mathbb{N}$, to prove *3* we only need to prove that it holds in intervals such as $(-m, m)$. We may also assume that $k > m$. Consider the following cases:

Case I. $-m < s \leq \frac{-1}{k}$ Here we have that

$$|f_k(s) - g(s)| = \left| -k \left[G\left(s - \frac{1}{k}\right) - G(s) \right] - g(s) \right| = \left| \frac{\left[G(s - \frac{1}{k}) - G(s) \right]}{\frac{-1}{k}} - g(s) \right|.$$

Then, by the Fundamental Theorem of Calculus we conclude that $f_k \to g$ uniformly.

Case II. $\frac{-1}{k} \leq s \leq 0$

Since $g(0) = 0$ and g is continuous, given $\epsilon > 0$ there exists $\delta > 0$ such that $|t| < \delta$ implies $|g(t)| < \epsilon/2$. Let $k_0 \in \mathbb{N}$ be such that $k_0 > m$ and $k_0 > 2/\delta$. Then, for $k > k_0$

$$\begin{aligned} |f_k(s) - g(s)| &= \left| k^2 s \left[G\left(\frac{-2}{k}\right) - G\left(\frac{-1}{k}\right) \right] - g(s) \right| \\ &\leq k^2 |s| \left| \int_{-1/k}^{-2/k} |g(t)| dt \right| + |g(s)| \\ &\leq k^2 (\frac{1}{k^2}) \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \quad \forall s \in [\frac{-1}{k}, 0]. \end{aligned}$$

Proving the desired convergence.

For the cases $0 \leq s \leq \frac{1}{k}$ and $\frac{1}{k} \leq s < m$ the arguments are similar.

□

The next theorem is our comparison principle. It will be useful to guarantee that, after applying the Galerkin Method, our candidate solution to problem (P_n) won't be the identical null solution. See Proposition 31 for information about its usage.

Theorem 19 (Theorem 3.1 from [1]). *Let $\psi : (0, +\infty) \to \mathbb{R}$ be a continuous function such that the mapping $(0, +\infty) \ni s \mapsto \frac{\psi(s)}{s}$ is strictly decreasing and $\rho > 0$. Suppose that there exist* even functions $v, w \in C^2(-\rho, \rho) \cap C[-\rho, \rho]$ *such that:*

1. $(A(w)w')' - w + \psi(w) \leq 0 \leq (A(v)v') - v + \psi(v)$ *in* $(-\rho, \rho)$*;*
2. $v, w \geq 0$ *in* $(-\rho, \rho)$ *and* $v(\rho) \leq w(\rho)$*;*
3. $\{x \in (-\rho, \rho); v(x) = 0\}$ *and* $\{x \in (-\rho, \rho); w(x) = 0\}$ *have null measure in* $\mathbb{R}$*;*
4. $v' \cdot w' \geq 0$ *in* $(-\rho, \rho)$*;*
5. $v', w' \in L^\infty(-\rho, \rho)$.

Then $v \leq w$ *in* $(-\rho, \rho)$.

See [1, Pag. 2419, Thm 3.1] for the proof; although slight different statement, defining $\frac{\psi(w(s))}{w(s)} = 0$ for $s \in \{x \in (-\rho, \rho); w(x) = 0\}$ is sufficient to completely adapt the demonstration gave in [1] for the above formulation.

4.2 APPROXIMATE PROBLEM

Let $\psi \in L^2(-n, n)$ be an *even and positive function.* In order to solve (P_n) we will focus our attention on the approximate problem

$$\begin{cases} -(A(u)u')'(t) + u(t) = \lambda a_1(t)|u(t)|^{q-1} + |u(t)|^{p-1} + f_k(|u'(t)|) + \frac{\psi}{k}, & \text{in } (-n, n) \\ u(n) = u(-n) = 0. \end{cases} \tag{P_n^k}$$

which carries significant information about (P_n), as result of the properties of f_k. This approach aims at constructing a sequence of solutions for (P_n^k) that will, eventually, converge to a solution of (P_n). In order to obtain such sequence, we will use the Galerkin method together with the following Lemma 20

Remark 8. We observe that the usage of the Strauss Approximation on a term involving u' is a novelty of our work.

Lemma 20. *Let $\mathfrak{F} : \mathbb{R}^N \to \mathbb{R}^N$ be a continuous function such that $\langle \mathfrak{F}(x), x \rangle \geq 0$ for all $x \in \mathbb{R}^N$ with $\|x\|_{\mathbb{R}^N} = r$. Then there exist x_0 in the closed ball $B[0, r]$ such that $\mathfrak{F}(x_0) = 0$.*

Proof. See [15, Chap. 5, Thm. 5.2.5]. □

Lemma 21 (Lemma 2 from [6]). *Let $g : \mathbb{R} \to \mathbb{R}$ be a continuous function satisfying (4.2). Then the sequence f_k of Theorem 18 satisfies*

1. *For all* $k \in \mathbb{N}$, $0 \leq sf_k(s) \leq C_1|s|^\theta$ *for every* $|s| \geq \frac{1}{k}$*;*

2. *for all* $k \in \mathbb{N}$, $0 \leq sf_k(s) \leq C_1|s|^2$ *for every* $|s| \leq \frac{1}{k}$*.*

where C_1 *is a constant independent of* k*.*

For the proof of this result see [6, Pag. 8, Lemma 2].

Definition 22. A function $w \in H_0^1(-n,n)$ is called an *E-weak solution* of (P_n^k) if w is an *even function* satisfying

$$\int_{-n}^{n} A(w)w'\varphi' + \int_{-n}^{n} w\varphi = \int_{-n}^{n} \lambda a_1|w|^{q-1}\varphi + \int_{-n}^{n} |w|^{p-1}\varphi + \int_{-n}^{n} f_k(|w'|)\varphi + \int_{-n}^{n} \frac{\psi}{k}\varphi$$

for all $\varphi \in \mathbb{E}_0^1(-n,n) = \{u \in H_0^1(-n,n); u(t) = u(-t) \text{ a.e } \}$.

Definition 23. We will call $w \in H_0^1(-n,n)$ a *weak solution* of (P_n^k) if

$$\int_{-n}^{n} A(w)w'v' + \int_{-n}^{n} wv = \int_{-n}^{n} \lambda a_1|w|^{q-1}v + \int_{-n}^{n} |w|^{p-1}v + \int_{-n}^{n} f_k(|w'|)v + \int_{-n}^{n} \frac{\psi}{k}v$$

for all $v \in H_0^1(-n,n)$.

Lemma 24 (Lemma 4.1 from [1])**.** *Let* $w \in H_0^1(-n,n)$ *be an E-weak solution of* (P_n^k)*. Then* w *is a weak solution of* (P_n^k)*.*

Proof. First we will prove that w is a weak solution in $(-n,0) \cup (0,n)$. Let $\zeta \in C_0^\infty(0,n)$ and define:

$$\overline{\zeta(t)} = \begin{cases} \zeta(t), & \text{if } t \in (0,n); \\ 0, & \text{if } t = 0; \\ \zeta(-t), & \text{if } t \in (-n,0). \end{cases}$$

Thus $\overline{\zeta} \in \mathbb{E}_0^1(-n,n)$ and

$$\int_{-n}^{n} A(w)w'\overline{\zeta}' + \int_{-n}^{n} w\overline{\zeta} = \int_{-n}^{n} \lambda a_1|w|^{q-1}\overline{\zeta} + \int_{-n}^{n} |w|^{p-1}\overline{\zeta} + \int_{-n}^{n} f_k(|w'|)\overline{\zeta} + \int_{-n}^{n} \frac{\psi}{k}\overline{\zeta}.$$

Since w, ψ, a_1 and $\overline{\zeta}$ are even functions, all the integrands above are also even functions; so the above equality can be rewritten as

$$\int_{0}^{n} A(w)w'\overline{\zeta}' + \int_{0}^{n} w\overline{\zeta} = \int_{0}^{n} \lambda a_1|w|^{q-1}\overline{\zeta} + \int_{0}^{n} |w|^{p-1}\overline{\zeta} + \int_{0}^{n} f_k(|w'|)\overline{\zeta} + \int_{0}^{n} \frac{\psi}{k}\overline{\zeta}.$$

Using that $\overline{\zeta}|_{(0,n)} = \zeta$, from the above equality one can conclude that w is a weak solution in $(0,n)$. Similarly we also obtain that w is a weak solution in $(-n,0)$.

Given $\varphi \in H_0^1(-n,n)$, let $\tilde{\psi} \in \mathbb{E}_0^1(-n,n)$ be such that $\tilde{\psi}(0) = \varphi(0)$. Defining

$$\Phi_1(t) = \begin{cases} \varphi(t) - \tilde{\psi}(t), & \text{if } t \in [0,n) \\ 0, & \text{if } t \in (-n,0] \end{cases}$$

$$\Phi_2(t) = \begin{cases} 0, & \text{if } t \in [0, n) \\ \varphi(t) - \tilde{\psi}(t), & \text{if } t \in (-n, 0] \end{cases}$$

we have that $\widetilde{\Phi_1} := \Phi_1|_{(0,n)} \in H_0^1(0,n)$ and $\widetilde{\Phi_2} := \Phi_2|_{(-n,0)} \in H_0^1(-n,0)$. Thus

$$\int_0^n A(w)w'\widetilde{\Phi_1}' + \int_0^n w\widetilde{\Phi_1} = \int_0^n \lambda a_1|w|^{q-1}\widetilde{\Phi_1} + \int_0^n |w|^{p-1}\widetilde{\Phi_1} + \int_0^n f_k(|w'|)\widetilde{\Phi_1} + \int_0^n \frac{\psi}{k}\widetilde{\Phi_1}. \quad (4.4)$$

$$\int_{-n}^0 A(w)w'\widetilde{\Phi_2}' + \int_{-n}^0 w\widetilde{\Phi_2} = \int_{-n}^0 \lambda a_1|w|^{q-1}\widetilde{\Phi_2} + \int_{-n}^0 |w|^{p-1}\widetilde{\Phi_2} + \int_{-n}^0 f_k(|w'|)\widetilde{\Phi_2} + \int_{-n}^0 \frac{\psi}{k}\widetilde{\Phi_2}. \quad (4.5)$$

Since $\tilde{\psi} \in \mathbb{E}_0^1(-n,n)$ we also have that

$$\int_{-n}^n A(w)w'\tilde{\psi}' + \int_{-n}^n w\tilde{\psi} = \int_{-n}^n \lambda a_1|w|^{q-1}\tilde{\psi} + \int_{-n}^n |w|^{p-1}\tilde{\psi} + \int_{-n}^n f_k(|w'|)\tilde{\psi} + \int_{-n}^n \frac{\psi}{k}\tilde{\psi}. \quad (4.6)$$

Combining (4.4),(4.5) and (4.6) we conclude

$$\int_{-n}^n A(w)w'\varphi' + \int_{-n}^n w\varphi = \int_{-n}^n \lambda a_1|w|^{q-1}\varphi + \int_{-n}^n |w|^{p-1}\varphi + \int_{-n}^n f_k(|w'|)\varphi + \int_{-n}^n \frac{\psi}{k}\varphi, \quad (4.7)$$

i.e, w is a weak solution. □

Proposition 25. *Properties of* $E_0^1(-n,n)$ *:*

(I) it is a Hilbert Space;

(II) it is separable;

(III) is has an orthonormal basis.

Proof. See Appendix A. □

Remark 9. For information concerning orthonormal basis in Hilbert spaces, see [16].

Let $\mathbb{E}_0^1(-n,n) = \{u \in H_0^1(-n,n); u(t) = u(-t) \text{ a.e } \}$ and $(e_l)_{l=1}^{\infty}$ be an orthonormal basis of $\mathbb{E}_0^1(-n,n)$.

Define $V_M = span\{e_1, \dots, e_M\}$; then for every $u \in V_M$ there exist $\xi_1, \dots, \xi_M$ in $\mathbb{R}$ such that $u = \sum_{i=1}^M \xi_i e_i$. By means of $T : V_M \to \mathbb{R}^M$, $T(u) = T(\sum_{i=1}^M \xi_i e_i) = (\xi_1, \dots, \xi_M)$, which is a linear isomorphism and preserve norm, we may define $\mathfrak{F} : \mathbb{R}^M \to \mathbb{R}^M$ such that

$$\mathfrak{F}(\xi) = (\mathfrak{F}_1(\xi), \dots, \mathfrak{F}_M(\xi)) \quad (4.8)$$

and

$$\mathfrak{F}_j(\xi) = \int_{-n}^n A(u)u'e_j' + \int_{-n}^n ue_j - \int_{-n}^n \lambda a_1|u|^{q-1}e_j - \int_{-n}^n |u|^{p-1}e_j - \int_{-n}^n f_k(|u'|)e_j - \int_{-n}^n \frac{\psi}{k}e_j$$

where $j \in \{1, \dots, M\}$ and $u = T^{-1}(\xi)$, for all $\xi \in \mathbb{R}^M$.

Lemma 26. *The function* $\mathfrak{F}$ *is continuous.*

Proof. Given $\xi = (\xi_1, \dots, \xi_M) \in \mathbb{R}^M$, let $(\xi_l)_{l=1}^{\infty}$ be a sequence in $\mathbb{R}^M$ such that $\|\xi_l - \xi\|_{\mathbb{R}^M} \to 0$. By means of T we can identify $T^{-1}(\xi) = u = \sum_{i=1}^M e_i\xi_i$ and $T^{-1}(\xi_l) = u_l = \sum_{i=1}^M e_i\xi_i^l$. Since T is isometry we have that $\|u_l - u\|_{W^{1,2}} \to 0$. That is, $\|u_l - u\|_{L^2} \to 0$ and $\|u_l' - u'\|_{L^2} \to 0$. Taking a subsequence if necessary, we may assume that

$$
\begin{aligned}
u_l(x) &\to u(x) \text{ a.e on } (-n, n), \\
u_l'(x) &\to u'(x) \text{ a.e on } (-n, n),
\end{aligned}
$$

and $|u_l(x)| \leq h_1(x)$, $|u_l'(x)| \leq h_2(x)$ a.e on $(-n, n)$, with $h_1, h_2 \in L^2(-n, n)$. Let $j \in \{1, 2, \dots, M\}$, we will prove that $f_j(\xi_l) \to f_j(\xi)$.

$$
\left|\int_{-n}^{n} A(u_l)u_l'e_j' - \int_{-n}^{n} A(u)u'e_j'\right| \leq \int_{-n}^{n} \left(|u_l'||A(u_l) - A(u)| + |A(u)||u_l' - u'|\right)|e_j'|, \tag{4.9}
$$

since $|u_l'(x)||A(u_l(x)) - A(u(x))||e_j'(x)| \to 0$ a.e and $|A(u(x))||u_l'(x) - u'(x)||e_j'(x)| \to 0$ a.e, by the Dominated Convergence Theorem (D.C.T) (4.9) tends to zero as $l \to +\infty$.

$$
\left|\int_{-n}^{n} u_le_j - \int_{-n}^{n} ue_j\right| \leq \int_{-n}^{n} |u_l - u||e_j| \to 0 \quad \text{[by (D.C.T).]} \tag{4.10}
$$

$$
\begin{aligned}
&\left|\int_{-n}^{n} [\lambda a_1(|u_l|^{q-1} - |u|^{q-1}) + (|u_l|^{p-1} - |u|^{p-1}) + (f_k(u_l') - f_k(u'))]e_j\right| \\
&\qquad \leq \int_{-n}^{n} \lambda|a_1|\left||u_l|^{q-1} - |u|^{q-1}\right||e_j| + \int_{-n}^{n} \left||u_l|^{p-1} - |u|^{p-1}\right||e_j| \\
&\qquad\qquad + \int_{-n}^{n} |f_k(|u_l'|) - f_k(|u'|)|\,|e_j|,
\end{aligned} \tag{4.11}
$$

since that $|u_l|^{q-1} \to |u|^{q-1}$ a.e and $|u_l|^{p-1} \to |u|^{p-1}$ a.e, (D.C.T) implies that the first two integrals above converges to zero. Using the second item of Theorem 18, we have

$$
\int_{-n}^{n} |f_k(|u_l'|) - f_k(|u'|)|\,|e_j| \leq \int_{-n}^{n} c(k)|u_l' - u'||e_j|. \tag{4.12}
$$

Then, by (D.C.T), (4.12) converges to 0 as $l \to +\infty$.

These estimations show us that for every subsequence (ξ_{l_k}) of (ξ_l), there exist a subsequence $(\xi_{l_{k_n}})$ of (ξ_{l_k}) that $\mathfrak{F}_j(\xi_{l_{k_n}}) \to \mathfrak{F}_j(\xi)$. Therefore $\mathfrak{F}_j(\xi_l) \to \mathfrak{F}_j(\xi)$. □

Proposition 27. *There exist $\lambda^* > 0$ and $k^* \in \mathbb{N}$ for which the problem (P_n^k) admits a nontrivial weak solution for every $\lambda \in (0, \lambda^*)$ and $k \geq k^*$.*

Proof. Our aim is to use Lemma 20, with the function $\mathfrak{F}$ defined in (4.8). Given $\xi \in \mathbb{R}^M$, we have that

$$
\langle \mathfrak{F}(\xi), \xi\rangle = \int_{-n}^{n} A(u)|u'|^2 + \int_{-n}^{n} |u|^2 - \int_{-n}^{n} \lambda a_1|u|^{q-1}u - \int_{-n}^{n} |u|^{p-1}u - \int_{-n}^{n} f_k(|u'|)u - \int_{-n}^{n} \frac{\psi}{k}u. \tag{4.13}
$$

In the following we will estimate these integrals. We have that

$$\int_{-n}^{n} \lambda a_1 |u|^{q-1} u \leq \lambda \|a_1\|_{L^s(\mathbb{R})} \|u\|_{L^2}^q \leq \lambda C_2 \|u\|_{W^{1,2}}^q, \tag{4.14}$$

$$\int_{-n}^{n} \frac{\psi}{k} u \leq \frac{\|\psi\|_{L^2(-n,n)} \|u\|_{W^{1,2}}}{k}. \tag{4.15}$$

Now let $\tilde{u} : \mathbb{R} \to \mathbb{R}$ be the extension by zero of u, then

$$\int_{-n}^{n} |u|^{p-1} u \leq \int_{-n}^{n} |u|^p = \int_{-n}^{n} |u|^2 |u|^{p-2} \tag{4.16}$$

$$\leq \|\tilde{u}\|_{L^\infty(\mathbb{R})}^{p-2} \int_{-n}^{n} |u|^2 \tag{4.17}$$

$$= \|\tilde{u}\|_{L^\infty(\mathbb{R})}^{p-2} \|u\|_{L^2}^2 \tag{4.18}$$

$$\leq C^{p-2} \|u\|_{W^{1,2}}^{p-2} \|u\|_{W^{1,2}}^2 = C^{p-2} \|u\|_{W^{1,2}}^p. \tag{4.19}$$

Where C is the constant for the embedding $W^{1,2}(\mathbb{R}) \hookrightarrow L^\infty(\mathbb{R})$.

Define

$$\Omega_> = \{s \in (-n,n); |u'(s)| \geq \frac{1}{k}\} \text{ and } \Omega_< = \{s \in (-n,n); 0 < |u'(s)| \leq \frac{1}{k}\}.$$

Then

$$\int_{-n}^{n} f_k(|u'|) u = \int_{\Omega_>} f_k(|u'|) u + \int_{\Omega_<} f_k(|u'|) u.$$

Notice that by Lemma 21,

$$\begin{aligned}
\int_{\Omega_<} f_k(|u'|) u &\leq \int_{\Omega_<} C_1 |u'||u| \leq \int_{\Omega_<} \frac{C_1}{k} |u| \\
&\leq \frac{C_1}{k} \int_{-n}^{n} |u| \leq \frac{C_1 (2n)^{1/2}}{k} \|u\|_{L^2} \\
&\leq \frac{C_1 (2n)^{1/2}}{k} \|u\|_{W^{1,2}}.
\end{aligned}$$

To estimate the integral over $\Omega_>$, consider the following cases :

Case 1. $2 < \theta < 3$.

Using Lemma 21, we have

$$\begin{aligned}
\int_{\Omega_>} f_k(|u'|) u &\leq \int_{\Omega_>} C_1 |u'|^{\theta-1} |u| \leq \int_{-n}^{n} C_1 |u'|^{\theta-1} |u| \\
&\leq C_1 \left(\int_{-n}^{n} |u|^w \right)^{\frac{1}{w}} \left(\int_{-n}^{n} |u'|^2 \right)^{\frac{\theta-1}{2}} \\
&\leq C_1 \left(\int_{\mathbb{R}} |\tilde{u}|^2 |\tilde{u}|^{w-2} \right)^{\frac{1}{w}} \|u'\|_{L^2}^{\theta-1} \\
&\leq C_1 \|\tilde{u}\|_{L^\infty(\mathbb{R})}^{\frac{w-2}{w}} \|u\|_{L^2}^{\frac{2}{w}} \|u'\|_{L^2}^{\theta-1} \\
&\leq C_1 C^{\frac{w-2}{w}} \|u\|_{W^{1,2}}^{\frac{w-2}{w}} \|u\|_{W^{1,2}}^{\frac{2}{w}} \|u\|_{W^{1,2}}^{\theta-1} = C_1 C^{\frac{w-2}{w}} \|u\|_{W^{1,2}}^{\theta}.
\end{aligned}$$

Where $w = \left(\frac{2}{\theta-1}\right)' = \frac{2}{3-\theta} > 2$.
Case 2. $\theta = 3$.

$$\begin{aligned}\int_{\Omega_>} f_k(|u'|)u \le \int_{\Omega_>} C_1|u'|^2|u| &\le \int_{-n}^{n} C_1|u'|^2|u| \\ &\le C_1\|\tilde{u}\|_{L^\infty(\mathbb{R})}\|u'\|_{L^2}^2 \le C_1C\|u\|_{W^{1,2}}\|u\|_{W^{1,2}}^2 \\ &= C_1C\|u\|_{W^{1,2}}^3.\end{aligned}$$

By the exposed necessity to consider the cases $2 < \theta < 3$ and $\theta = 3$, we shall estimate (4.13) in two cases as well.
Case 1. $2 < \theta < 3$.

$$\begin{aligned}\langle \mathfrak{F}(\xi), \xi\rangle \ge &\gamma\|u\|_{W^{1,2}}^2 - \lambda C_2\|u\|_{W^{1,2}}^q - C^{p-2}\|u\|_{W^{1,2}}^p \\ &- C_1C^{\frac{w-2}{w}}\|u\|_{W^{1,2}}^\theta - \left(\frac{C_1(2n)^{1/2}}{k} + \frac{\|\psi\|_{L^2(-n,n)}}{k}\right)\|u\|_{W^{1,2}}.\end{aligned}$$

Define $Z_k : \mathbb{R}^+ \to \mathbb{R}$ by

$$Z_k(x) = \gamma x^2 - \lambda C_2 x^q - C^{p-2}x^p - C_1C^{\frac{w-2}{w}}x^\theta - \left(\frac{C_1(2n)^{1/2}}{k} + \frac{\|\psi\|_{L^2(-n,n)}}{k}\right)x.$$

We would like to find $x_1 \in \mathbb{R}^+_*$ such that

$$\gamma x_1^2 - C^{p-2}x_1^p - C_1C^{\frac{w-2}{w}}x_1^\theta > \frac{x_1^2}{2}\gamma \tag{4.20}$$

or equivalently,

$$\frac{\gamma}{2} > C^{p-2}x_1^{p-2} + C_1C^{\frac{w-2}{w}}x_1^{\theta-2}.$$

For this, if we take

$$\delta_1 = \min\left\{\left(\frac{\gamma}{4C^{p-2}}\right)^{1/(p-2)}, \left(\frac{\gamma}{4C_1C^{\frac{w-2}{w}}}\right)^{1/(\theta-2)}\right\},$$

then for $0 < x_1 < \delta_1$ (4.20) is true. Consequently,

$$Z_k(x_1) \ge \frac{x_1^2}{2}\gamma - \lambda C_2\delta_1^q - \left(\frac{C_1(2n)^{1/2}}{k} + \frac{\|\psi\|_{L^2(-n,n)}}{k}\right)\delta_1.$$

Define $\rho_1 = \frac{x_1^2}{2}\gamma - \lambda C_2\delta_1^q$. We will adjust $\lambda > 0$ so that $\rho_1 > 0$; for this if $\rho_1 > 0$ it would imply that

$$\frac{x_1^2}{2}\gamma - \lambda C_2\delta_1^q > 0 \Leftrightarrow \frac{x_1^2\gamma}{2C_2\delta_1^q} > \lambda.$$

Take $\Lambda_1 = \frac{x_1^2\gamma}{2C_2\delta_1^q}$ and $0 < \lambda < \Lambda_1$. Thus $\rho_1 > 0$ and we can find $k_1 \in \mathbb{N}$ such that for $k > k_1$, $\rho_1 > \left(\frac{C_1(2n)^{1/2}}{k} + \frac{\|\psi\|_{L^2(-n,n)}}{k}\right)\delta_1 > 0$. Therefore, for $0 < x_1 < \delta_1$, $0 < \lambda < \Lambda_1$ and $k > k_1$

$$Z_k(x_1) > 0,$$

and so, with $\|u\|_{W^{1,2}} = x_1$,

$$\langle \mathfrak{F}(\xi), \xi\rangle > 0. \tag{4.21}$$

Case 2. $\theta = 3$.

$$\begin{aligned}\langle \mathfrak{F}(\xi), \xi \rangle \geq & \gamma \|u\|_{W^{1,2}}^2 - \lambda C_2 \|u\|_{W^{1,2}}^q - C^{p-2} \|u\|_{W^{1,2}}^p \\ & - C_1 C \|u\|_{W^{1,2}}^3 - \left(\frac{C_1 (2n)^{1/2}}{k} + \frac{\|\psi\|_{L^2(-n,n)}}{k} \right) \|u\|_{W^{1,2}}.\end{aligned}$$

Following the same ideas as in **Case 1**, take

$$\delta_2 = \min \left\{ \left(\frac{\gamma}{4C^{p-2}} \right)^{1/(p-2)}, \left(\frac{\gamma}{4C_1 C} \right) \right\}.$$

For $0 < x_2 < \delta_2$ and $\rho_2 = \frac{x_2^2 \gamma}{2} - \lambda C_2 \delta_2^q$, with $0 < \lambda < \frac{x_2^2 \gamma}{2C_2 \delta_2^q} := \Lambda_2$, we have $\rho_2 > 0$. Let $k_2 \in \mathbb{N}$ be such that $k > k_2 \Rightarrow \rho_2 > \left(\frac{C_1 (2n)^{1/2}}{k} + \frac{\|\psi\|_{L^2(-n,n)}}{k} \right) \delta_2 > 0$. Then, for $\|u\|_{W^{1,2}} = x_2$ we have

$$\langle \mathfrak{F}(\xi), \xi \rangle > 0. \tag{4.22}$$

To properly use Lemma 20, let us take $0 < r < \min\{\delta_1, \delta_2\}$, $0 < \lambda < \lambda^* := \min\{\Lambda_1, \Lambda_2\}$ and $k > k^* := \max\{k_1, k_2\}$. In this way, for $||u||_{W^{1,2}} = r$, we conclude that $\langle \mathfrak{F}(\xi), \xi \rangle > 0$ independent of θ. By Lemma 20, there exists $y_M \in B[0, r]$ such that $\mathfrak{F}(y_M) = 0$ that is, identifying $v_M = T^{-1}(y_M)$, for all $j \in \{1, \ldots, M\}$

$$\begin{aligned}&\int_{-n}^n A(v_M) v_M' e_j' + \int_{-n}^n v_M e_j = \qquad (4.23)\\ &= \int_{-n}^n \lambda a_1 |v_M|^{q-1} e_j + \int_{-n}^n |v_M|^{p-1} e_j + \int_{-n}^n f_k(|v_M'|) e_j + \int_{-n}^n \frac{\psi}{k} e_j.\end{aligned}$$

Therefore (4.23) holds for all $\varphi \in V_M$, because $\{e_1, \ldots, e_M\}$ is a basis of V_M.

Remark 10. Notice that our choice of r does not depend on M,n or k. This free determination of r will be useful further down in the argumentation, because using the embedding $W^{1,2}(\mathbb{R}) \hookrightarrow L^\infty(\mathbb{R})$ we will be able to obtain an uniform upper bound, in the norm of $L^\infty(\mathbb{R})$, for the sequence of solutions of the problem (P_n^k). Then this upper bound will naturally be transferred to also bound the sequence of solution of (P_n).

Since $\|v_M\|_{W^{1,2}} \leq r$ there is $v_0 \in \mathbb{E}_0^1(-n, n)$ such that $v_M \rightharpoonup v_0$ in $H_0^1(-n, n)$. By the compact embedding $W^{1,2}(-n, n) \hookrightarrow L^2(-n, n)$ we conclude $v_M \to v_0$ in $L^2(-n, n)$. Our goal is to show that v_0 is a weak solution of (P_n^k). Let $\Gamma_M : V_M \to V_M^*$ and $B_M : V_M \to V_M^*$ be defined by

$$\langle \Gamma_M(v), \varphi \rangle = \int_{-n}^n A(v) v' \varphi' \tag{4.24}$$

and (4.25)

$$\langle B_M(v), \varphi \rangle = \int_{-n}^n \left(-v + \lambda |v|^{q-1} + |v|^{p-1} + f_k(|v'|) + \frac{\psi}{k} \right) \varphi.$$

Hence, $\langle \Gamma_M(v_M) - B_M(v_M), \varphi\rangle = 0$ for all $\varphi \in V_M$.

Denoting $P_M : \mathbb{E}_0^1(-n,n) \to V_M$ the projection of $\mathbb{E}_0^1(-n,n)$ onto V_M,(that is, if $u = \sum_{i=1}^{\infty} \alpha_i e_i$ then $P_M(u) = \sum_{i=1}^{M} \alpha_i e_i$) we have

$$\langle \Gamma_M(v_M) - B_M(v_M), v_M - P_M v_0\rangle = 0,$$

so

$$\begin{aligned} &\langle \Gamma_M(v_M), v_M - P_M v_0\rangle = \langle B_M(v_M), v_M - P_M v_0\rangle = \\ &= \int_{-n}^{n} \left(-v_M + \lambda |v_M|^{q-1} + |v_M|^{p-1} + f_k(|v_M'|) + \frac{\psi}{k} \right) (v_M - P_M v_0). \end{aligned} \tag{4.26}$$

Letting $M \to \infty$ one can see without difficulties that $\langle \Gamma_M(v_M), v_M - P_M v_0\rangle \to 0$ (see Appendix A). This convergence allows us to prove the following

Lemma 28. *$v_M \to v_0$ strongly, i.e in the norm of $H_0^1(-n,n)$.*

Remark 11. The idea to consider the operators Γ_M and B_M was an inspiration from the arguments presented in [17].

Proof. The limit $\|v_M - v_0\|_{L^2(-n,n)} \to 0$ has been established before,thus we will focus our efforts demonstrating the same for $\|v_M' - v_0'\|_{L^2(-n,n)}$. Let $\Phi_M, \Phi, \Psi_M, \zeta_M \in (\mathbb{E}_0^1(-n,n))^*$ be given by

$$\Phi_M(w) = \int_{-n}^{n} A(v_M) v_0' w' \tag{4.27}$$

$$\Phi(w) = \int_{-n}^{n} A(v_0) v_0' w' \tag{4.28}$$

$$\Psi_M(w) = \int_{-n}^{n} A(v_M) P_M v_0' w' \tag{4.29}$$

$$\zeta_M(w) = \int_{-n}^{n} A(v_0) P_M v_0' w'. \tag{4.30}$$

Then, by a straightforward calculation, (see Appendix A), $|\Phi_M - \Phi| \to 0$,$|\Psi_M - \Phi_M| \to 0$ and $|\zeta_M - \Phi| \to 0$ in $(\mathbb{E}_0^1(-n,n))^*$. Thus $|\Psi_M - \zeta_M| \to 0$ in $(\mathbb{E}_0^1(-n,n))^*$, since $|\Psi_M - \zeta_M| \le |\Psi_M - \Phi_M| + |\Phi_M - \Phi| + |\Phi - \zeta_M|$. Writing $\Psi_M = (\Psi_M - \zeta_M) + \zeta_M$ yields that $\Psi_M \to \Phi$ in $(\mathbb{E}_0^1(-n,n))^*$. Remembering the weak convergence $v_M \rightharpoonup v_0$ one can conclude $(v_M - P_M v_0) \rightharpoonup 0$ in $\mathbb{E}_0^1(-n,n)$ because for all $f \in (\mathbb{E}_0^1(-n,n))^*$

$$|f(v_M) - f(P_M v_0)| \le |f(v_M) - f(v_0)| + \|f\| \|v_0 - P_M v_0\|_{W^{1,2}}.$$

Consequently, letting $M \to \infty$, $\Psi_M(v_M - P_M v_0) \to \Phi(0) = 0$. This means that

$$\int_{-n}^{n} A(v_M) P_M v_0' (v_M' - P_M v_0') \to 0. \tag{4.31}$$

Also, rewriting (4.26)

$$\int_{-n}^{n} A(v_M) v_M' (v_M' - P_M v_0') \to 0 \text{ as } M \to \infty. \tag{4.32}$$

Therefore, from (4.32)–(4.31)

$$\int_{-n}^{n} A(v_M)(v_M' - P_M v_0')^2 \to 0 \text{ as } M \to \infty. \tag{4.33}$$

Since $A(x) \geq \gamma > 0$ for all $x \in \mathbb{R}$ we conclude $\|v_M' - P_M v_0'\|_{L^2(-n,n)} \to 0$ as $M \to \infty$. Then $\|v_M' - v_0'\|_{L^2(-n,n)} \to 0$ as result of $\|v_M' - v_0'\|_{L^2(-n,n)} \leq \|v_M' - P_M v_0'\|_{L^2(-n,n)} + \|v_0' - P_M v_0'\|_{L^2(-n,n)}$, proving the Lemma. □

We know that for every $\varphi \in V_M$

$$\begin{aligned} &\int_{-n}^{n} A(v_M) v_M' \varphi' + \int_{-n}^{n} v_M \varphi = \\ &= \int_{-n}^{n} \lambda a_1 |v_M|^{q-1} \varphi + \int_{-n}^{n} |v_M|^{p-1} \varphi + \int_{-n}^{n} f_k(|v_M'|)\varphi + \int_{-n}^{n} \frac{\psi}{k} \varphi. \end{aligned} \tag{4.34}$$

By the previous lemma, taking a subsequence if necessary, we may assume that $v_M'(x)$ converges a.e to $v_0'(x)$ and there exists $h \in L^2(-n,n)$ such that $|v_M'(x)| \leq h(x)$ a.e. Then notice that

$$\left| \int_{-n}^{n} \left(A(v_M)v_M' - A(v_0)v_0'\right) \varphi' \right| \leq \left(\int_{-n}^{n} |A(v_M)v_M' - A(v_0)v_0'|^2 \right)^{1/2} \|\varphi'\|_{L^2} \tag{4.35}$$

and exists $Q > 0$ such that $\|v_M\|_\infty < Q$ for all $M \in \mathbb{N}$, because v_M converges to v_0 in $C^0[-n,n]$ due to the embedding $W^{1,2}(-n,n) \hookrightarrow C^0[-n,n]$. We can suppose Q big enough so that $Q > \max\{\|v_0\|_\infty + A(0), \|v_M\|_\infty + A(0)\}$. Since

$$|A(v_M(x))v_M'(x) - A(v_0(x))v_0'(x)| \to 0 \ \text{ a.e} \tag{4.36}$$

and

$$\begin{aligned} |A(v_M(x))v_M'(x) - A(v_0(x))v_0'(x)|^2 &\leq \left(|A(v_M(x))v_M'(x)| + |A(v_0(x))v_0'(x)|\right)^2 \\ &\leq |A(v_M(x))|^2 |v_M'(x)|^2 \\ &\quad + 2|A(v_M(x))||A(v_0(x))||v_M'(x)||v_0'(x)| \\ &\quad + |A(v_0(x))|^2 |v_0'(x)|^2 \\ &\leq \tilde{A}^2 Q^2 h^2(x) + 2\tilde{A}^2 Q^2 |v_0'(x)| h(x) + \tilde{A}^2 Q^2 |v_0'(x)|^2 \end{aligned}$$

almost everywhere, we conclude by (D.C.T) that

$$\int_{-n}^{n} A(v_M) v_M' \varphi' \to \int_{-n}^{n} A(v_0) v_0' \varphi' \quad \text{as } M \to \infty. \tag{4.37}$$

Also, by direct calculation, the following convergences are true

$$\int_{-n}^{n} v_M \varphi \to \int_{-n}^{n} v_0 \varphi \tag{4.38}$$

$$\int_{-n}^{n} \lambda a_1 |v_M|^{q-1} \varphi \to \int_{-n}^{n} \lambda a_1 |v_0|^{q-1} \varphi \tag{4.39}$$

$$\int_{-n}^{n} |v_M|^{p-1} \varphi \to \int_{-n}^{n} |v_0|^{p-1} \varphi \tag{4.40}$$

$$\int_{-n}^{n} f_k(|v_M'|)\varphi \to \int_{-n}^{n} f_k(|v_0'|)\varphi \tag{4.41}$$

as $M \to \infty$. Thus, for every $\varphi \in V_M$

$$\begin{aligned} &\int_{-n}^{n} A(v_0)v_0'\varphi' + \int_{-n}^{n} v_0\varphi = \\ &= \int_{-n}^{n} \lambda a_1|v_0|^{q-1}\varphi + \int_{-n}^{n} |v_0|^{p-1}\varphi + \int_{-n}^{n} f_k(|v_0'|)\varphi + \int_{-n}^{n} \frac{\psi}{k}\varphi. \end{aligned} \tag{4.42}$$

Furthermore, for every $u \in \mathbb{E}_0^1(-n,n)$, it follows that

$$\int_{-n}^{n} A(v_0)v_0'P_M u' \to \int_{-n}^{n} A(v_0)v_0'u' \tag{4.43}$$

$$\int_{-n}^{n} v_0 P_M u \to \int_{-n}^{n} v_0 u \tag{4.44}$$

$$\int_{-n}^{n} \lambda a_1|v_0|^{q-1} P_M u \to \int_{-n}^{n} \lambda a_1|v_0|^{q-1} u \tag{4.45}$$

$$\int_{-n}^{n} |v_0|^{p-1} P_M u \to \int_{-n}^{n} |v_0|^{p-1} u \tag{4.46}$$

$$\int_{-n}^{n} f_k(|v_0'|) P_M u \to \int_{-n}^{n} f_k(|v_0'|) u \tag{4.47}$$

as $M \to \infty$. Thus, for every $u \in \mathbb{E}_0^1(-n,n)$

$$\begin{aligned} &\int_{-n}^{n} A(v_0)v_0'u' + \int_{-n}^{n} v_0 u = \\ &= \int_{-n}^{n} \lambda a_1|v_0|^{q-1}u + \int_{-n}^{n} |v_0|^{p-1}u + \int_{-n}^{n} f_k(|v_0'|)u + \int_{-n}^{n} \frac{\psi}{k}u. \end{aligned} \tag{4.48}$$

So v_0 is an *E-weak solution* of (P_n^k); by Lemma 24 v_0 is also a weak solution. This finishes the proof of Proposition 27. □

In what follows we will make $k \to \infty$ thus we can consider $\psi \equiv 1$, because the term $\frac{\psi}{k}$ will converge to 0 as $k \to \infty$.

Proposition 29. *The above weak solution v_0 satisfies:*

1. $v_0 \in C^{1,\beta}[-n,n] \cap C^2(-n,n)$;
2. $v_0(t) \geq 0$.

Proof. To prove *1*, we will use the Theorem 1 of [7, Pag. 1]. Let $F : [-n,n] \times [-Cr, Cr] \times \mathbb{R} \to \mathbb{R}$ be defined by $F(x,z,p) = A(z)p$, where C is the embedding constant for $W^{1,2}(\mathbb{R}) \hookrightarrow L^\infty(\mathbb{R})$, and $B(x,z,p) = z - (\lambda a_1(x)|z|^{q-1} + |z|^{p-1} + f_k(|p|) + \frac{1}{k})$ be defined in the same domain. Then, problem (P_n^k) may be rewritten as

$$div_x F(x, u(x), u'(x)) + B(x, u(x), u'(x)) = 0.$$

In order to use Theorem 1 of [7, Pag. 1] we must verify the existence of nonnegative constants l, L, M_0, m, κ with $l \leq L$ such that

$$\frac{\partial F}{\partial p}(x,z,p)\xi^2 \geq l(\kappa + |p|)^m \xi^2, \tag{4.49}$$

$$\left|\frac{\partial F}{\partial p}(x,z,p)\right| \leq L(\kappa + |p|)^m, \tag{4.50}$$

$$|F(x,z,p) - F(y,w,p)| \leq L(1+|p|)^{m+1} \cdot |z-w|, \tag{4.51}$$

$$|B(x,z,p)| \leq L(1+|p|)^{m+2} \tag{4.52}$$

for all $(x,z,p) \in \{-n,n\} \times [-M_0, M_0] \times \mathbb{R}$, $w \in [-M_0, M_0]$ and $\xi \in \mathbb{R}$. Since $\frac{\partial F}{\partial p}(x,z,p) = A(z)$, inequality (4.49) follows from $A(z)\xi^2 \geq \gamma\xi^2$, that is, $l = \gamma$.

To prove the remaining inequalities take $M_0 = Cr$,

$$T > \max\{Cr + \lambda \max\{|a_1(-n)|, |a_1(n)|\}|Cr|^{q-1} + |Cr|^{p-1} + 1, 2C_1, A(Cr), \tilde{A}\},$$

$L = 2T, \kappa = 0$ and $m = 0$, where $\tilde{A}$ is the Lipchitz constant of A. Then :

(4.50)

$$\left|\frac{\partial F}{\partial p}(x,z,p)\right| = A(z) \leq A(Cr) < L;$$

(4.51)

$$|F(x,z,p) - F(y,w,p)| = |A(z)p - A(w)p| \leq \tilde{A}|p||z-w| \leq L(1+|p|)|z-w|;$$

(4.52)

$$\begin{aligned} |B(x,z,p)| &= |z - (\lambda a_1(x)|z|^{q-1} + |z|^{p-1} + f_k(|p|) + \frac{1}{k})| \\ &\leq Cr + \lambda \max\{|a_1(-n)|, |a_1(n)|\}|Cr|^{q-1} + |Cr|^{p-1} + 1/k + C_1(1+|p|^{\theta-1}) \\ &\leq T + C_1(1 + (1+|p|)^{\theta-1}) \\ &\leq T + 2C_1(1+|p|)^2 \\ &\leq T(1 + (1+|p|)^2) \\ &\leq 2T(1+|p|)^2 = L(1+|p|)^2. \end{aligned} \tag{4.53}$$

Therefore, by Theorem 1 of [7] there exists $\beta \in (0,1)$ and a constant $\hat{C}$, independent of k, such that $v_0 \in C^{1,\beta}([-n,n])$ and

$$|v_0|_{1+\beta} \leq \hat{C}. \tag{4.54}$$

It follows from [18, pag. 317, Chap. 6, Thm. 4] that $v_0 \in W^{2,2}(-n,n)$ and since v_0 is a weak solution of (P_n^k) we have

$$v_0'' = \frac{v_0 - \lambda a_1|v_0|^{q-1} - |v_0|^{p-1} - f_k(|v_0'|) - 1/k - A'(v_0)|v_0'|^2}{A(v_0)} \tag{4.55}$$

showing that v_0'' is continuous, thus $v_0 \in C^2(-n,n)$.

To prove that $v_0(t) \geq 0$ for all $t \in (-n,n)$ we first notice that $v_0^-(t) = \max\{0, -v_0(t)\} \in H_0^1(-n,n)$. We will borrow the argument presented in [6, Pag. 14]. Using $v_0^-(t)$ as a test function in the definition of weak solution provides

$$-\int_{-n}^{n} A(v_0)|v_0^-|^2 - \int_{-n}^{n} |v_0^-|^2 = \int_{-n}^{n} \lambda a_1 |v_0|^{q-1} v_0^- + \int_{-n}^{n} |v_0|^{p-1} v_0^- + \int_{-n}^{n} f_k(|v_0'|) v_0^- + \int_{-n}^{n} \frac{1}{k} v_0^- . \tag{4.56}$$

Then $-\gamma \|v_0^-\|_{W^{1,2}}^2 \geq 0$, thus $\|v_0^-\|_{W^{1,2}} = 0$ implying $v_0^- \equiv 0$ a.e. Since v_0 is continuous, $v_0(t) \geq 0$ for all $t \in (-n,n)$.

□

4.2.1 Constructing a Solution to Problem (P_n)

Let v_k be the (strong) solution of problem (P_n^k) – obtained just above – with k varying. By the previous constructions we have that $\|v_k\|_{W^{1,2}(-n,n)} \leq r$ independent of k, as noticed in Remark 10. Then there exists $u_n \in H_0^1(-n,n)$, $\|u_n\|_{W^{1,2}(-n,n)} \leq r$, so that v_k has a subsequence converging weakly in $H_0^1(-n,n)$ to u_n. From now on v_k will denote this subsequence. Since the function

$$H_0^1(-n,n) \ni w \mapsto \int_{-n}^{n} A(u_n) u_n' w'$$

belongs to $(H_0^1(-n,n))^*$, we have – by the weak convergence – that

$$\int_{-n}^{n} A(u_n) u_n' (v_k - u_n)' \to 0 \ \text{ as } k \to \infty.$$

This convergence will be useful in our next task: to prove that $v_k \to u_n$ strongly in $H_0^1(-n,n)$.

Lemma 30. *The following convergence is true*

$$\int_{-n}^{n} A(u_n) v_k' (v_k - u_n)' \to 0 \ \text{as } k \to \infty.$$

Proof. We might write

$$\begin{aligned}\int_{-n}^{n} A(u_n) v_k'(v_k - u_n)' &= \int_{-n}^{n} [A(u_n) - A(v_k) + A(v_k)] v_k' (v_k' - u_n') \\ &= \underbrace{\int_{-n}^{n} [A(u_n) - A(v_k)] v_k'(v_k' - u_n')}_{I_1} + \underbrace{\int_{-n}^{n} A(v_k) v_k'(v_k' - u_n')}_{I_2}\end{aligned}$$

and analyze I_1 and I_2 separately.

Analysis of I_2. By the weak formulation of (P_n^k)

$$\int_{-n}^{n} A(v_k) v_k' (v_k' - u_n') =$$

$$\begin{cases} \int_{-n}^{n} -v_k(v_k - u_n) + \lambda a_1 |v_k|^{q-1}(v_k - u_n) + |v_k|^{p-1}(v_k - u_n) \\ + \int_{-n}^{n} \frac{(v_k - u_n)}{k} \end{cases} \tag{4.57}$$

$$+ \int_{-n}^{n} f_k(|v_k'|)(v_k - u_n). \tag{4.58}$$

Since we have compact injection of $H_0^1(-n,n)$ onto $L^2(-n,n)$, the weak convergence of v_k to u_n in $H_0^1(-n,n)$ implies $\|v_k - u_n\|_{L^2} \to 0$. Thus it is straightforward to see that (4.57) converges to 0 as $k \to \infty$. Remains to verify what happen with (4.58) in the limit. We have that

$$\int_{-n}^{n} f_k(|v_k'|)(v_k - u_n) \leq \int_{-n}^{n} C_1(|v_k'|^{\theta-1} + |v_k'|)(v_k - u_n)$$

by Lemma 21. Using (4.54), that is, the estimation $|v_k|_{1,\beta} \leq \hat{C}$ which is independent of k, we have that

$$|v_k'|^{\theta-1} + |v_k'| \leq (\hat{C})^{\theta-1} + \hat{C}.$$

Then,

$$\int_{-n}^{n} f_k(|v_k'|)(v_k - u_n) \leq C_1[(\hat{C})^{\theta-1} + \hat{C}] \int_{-n}^{n} (v_k - u_n) \tag{4.59}$$

$$\leq \underbrace{(2n)^{1/2} C_1[(\hat{C})^{\theta-1} + \hat{C}]\|v_k - u_n\|_{L^2}}_{\to 0 \text{ as } k\to\infty}. \tag{4.60}$$

Thus, $\lim_{k\to\infty} I_2(k) = 0$.

Analysis of I_1. We also have that $\lim_{k\to\infty} I_1(k) = 0$, as one can see through

$$\begin{aligned}
\left|\int_{-n}^{n} [A(u_n) - A(v_k)]v_k'(v_k' - u_n')\right| &\leq \int_{-n}^{n} |A(u_n) - A(v_k)||v_k'||v_k' - u_n'| \\
&\leq \hat{C} \int_{-n}^{n} |A(u_n) - A(v_k)||v_k' - u_n'| \\
&\leq \hat{C}\tilde{A} \int_{-n}^{n} |u_n - v_k||v_k' - u_n'| \\
&\leq \hat{C}\tilde{A}\|u_n - v_k\|_{L^2}\|v_k' - u_n'\|_{L^2} \\
&\leq \hat{C}\tilde{A}2r\|u_n - v_k\|_{L^2}.
\end{aligned}$$

Proving the Lemma. □

Thus,

$$\int_{-n}^{n} A(u_n)u_n'(v_k - u_n)' \to 0 \quad \text{as } k \to \infty \tag{4.61}$$

$$\int_{-n}^{n} A(u_n)v_k'(v_k - u_n)' \to 0 \text{ as } k \to \infty. \tag{4.62}$$

From (4.62) – (4.61) we have

$$\int_{-n}^{n} A(u_n)(v_k' - u_n')^2 \to 0 \text{ as } k \to \infty \tag{4.63}$$

implying that $v_k' \to u_n'$ in $L^2(-n,n)$, since γ is a uniform lower-bound for A. Hence $v_k \to u_n$ in $H_0^1(-n,n)$. □

Remark 12. Since $v_k \to u_n$ in $H_0^1(-n,n)$ we conclude that u_n is also an even function; due to the embedding $W^{1,2}(-n,n) \hookrightarrow C[-n,n]$.

Proposition 31. *The function* u_n *satisfies:*

1. u_n *is strictly positive in* $(-n, n)$*;*

2. u_n *is a solution to* (P_n)*.*

Proof.

Item 1. Let $\tilde{a} := \inf_{x\in[-n,n]} a_1(x)$. We will divide our argument in two cases:

Remark 13. This division of cases is a geometric argument that we borrowed from [1].

Case 1. There exists a subsequence $(v_{k_i})_{i\in\mathbb{N}}$ of (v_k) such that $v'_{k_i} \geq 0$ in $(-n, 0)$ for all i.

Consider the problem

$$\begin{cases} -(A(u)u')' + u = \lambda\tilde{a}|u|^{q-1} & \text{in } (-n, n) \\ u > 0 & \text{in } (-n, n) \\ u(-n) = u(n) = 0. \end{cases} \tag{4.64}$$

Since $v'_{k_i} \geq 0$ in $(-n, 0)$ we get that $v_{k_i} > 0$ in $(-n, 0)$, because – due to Proposition 29 – v_{k_i} is an even solution of (P_n^k) and $v_{k_i} \geq 0$; i.e, supposing the existence of $x_i \in (-n, 0)$ such that $v_{k_i}(x_i) = 0$ implies the existence of $y_i \in (-n, 0)$ such that $v'_{k_i}(y_i) < 0$, which would be a contradiction. Thus we see that v_{k_i} is a sup-solution for this equation. Let ϕ_1 be an even and positive eigenfunction for the eigenvalue problem

$$\begin{cases} -u'' = \lambda_1 u & \text{in } (-n, n) \\ u(-n) = u(n) = 0 \end{cases} \tag{4.65}$$

where $\lambda_1 = \frac{\pi^2}{(2n)^2}$. Thus, choosing τ such that

$$\frac{\tau^{2-q}(1 + \gamma\lambda_1)}{\lambda\tilde{a}} \leq \phi_1^{q-2}$$

we have that $\tau\phi_1$ is as sub-solution of (4.64). By Theorem 19

$$v_{k_i}(t) \geq \tau\phi_1(t) \quad \forall t \in (-n, n),$$

therefore, in the limit,

$$u_n(t) \geq \tau\phi_1(t) > 0 \quad \forall t \in (-n, n).$$

Case 2. For all subsequence of (v_k) there exists a subsubsequence $(v_{k_i})_{i\in\mathbb{N}}$ and exist a sequence $(z_i)_{i\in\mathbb{N}} \subset (-n, 0)$ such that $v'_{k_i}(z_i) < 0$.

Remark 14. Although the geometric argument is an inspiration from [1], we still need to adjust it to our necessity. Lemma 32 is one of such adjustments.

Lemma 32. *If* $x \in (-n, n)$ *and* $v''_{k_i}(x) \geq 0$*, then* $v_{k_i}(x) > (\lambda\tilde{a})^{\frac{1}{2-q}}$*.*

Proof. Since v_{k_i} is a solution for the problem (P_n^k), with $k = k_i$, for all t

$$-A'(v_{k_i}(t))|v'_{k_i}(t)|^2 - A(v_{k_i}(t))v''_{k_i}(t) + v_{k_i}(t) \geq \lambda\tilde{a}|v_{k_i}(t)|^{q-1} + \frac{1}{k_i}. \tag{4.66}$$

Then, with $t = x$,

$$\begin{aligned}-A(v_{k_i}(x))v''_{k_i}(x) &> \lambda\tilde{a}|v_{k_i}(x)|^{q-1} - v_{k_i}(x) + A'(v_{k_i}(x))|v_{k_i}(x)|^2 \\ &\geq \lambda\tilde{a}|v_{k_i}(x)|^{q-1} - v_{k_i}(x).\end{aligned}$$

By hypotheses $v''_{k_i}(x) \geq 0$, then using the previous inequality,

$$v_{k_i}(x) > \lambda\tilde{a}|v_{k_i}(x)|^{q-1}.$$

thus $v_{k_i}(x) \neq 0$; also follows by the previous inequality that $v_{k_i}(x) > (\lambda\tilde{a})^{\frac{1}{2-q}}$. □

Now, in order to use this lemma, we ought to find a $x_i \in (-n, n)$ such that $v''_{k_i}(x_i) \geq 0$. Using the fact that v_{k_i} is even and $v'_{k_i}(z_i) < 0$ we have that $v'_{k_i}(-z_i) > 0$. Let $x_i = \min_{x \in [z_i, -z_i]} v_{k_i}(x)$ and notice that $x_i \neq z_i$ and $x_i \neq -z_i$; indeed, there exist $\delta > 0$ such that, if $x \in (z_i, z_i + \delta) \cup (-z_i - \delta, -z_i)$, then $v_{k_i}(x) < v_{k_i}(z_i) = v_{k_i}(-z_i)$. Hence $x_i \in (z_i, -z_i)$ and $v'_{k_i}(x_i) = 0$; therefore $v''_{k_i}(x_i)$ must be *greater or equal than 0*, because if $v''_{k_i}(x_i) < 0$ there would be $\xi > 0$ such that, for $x \in (x_i, x_i + \xi) \subset (z_i, -z_i)$, $v'_{k_i}(x) < 0$; and for this neighborhood $v_{k_i}(x) < v_{k_i}(x_i)$ – a contradiction with the minimality of x_i. Thus for all i

$$v_{k_i}(x_i) > (\lambda\tilde{a})^{\frac{1}{2-q}}.$$

From the compacity of $[-n, n]$, there exist $x_0 \in [-n, n]$ such that $x_i \to x_0$ when $i \to \infty$; taking a subsequence if necessary. Then

$$u_n(x_0) = \lim_{i\to\infty} v_{k_i}(x_i) \geq (\lambda\tilde{a})^{\frac{1}{2-q}} > 0.$$

Finally we will conclude *item 1* showing that, also in this case, u_n is strictly positive in $(-n, n)$.

Suppose by contradiction that there exists $y \in (-n, n)$ such that $u_n(y) = 0$. Let $(d, s) \subset (-n, n)$ be the biggest interval containing y satisfying the property: if $x \in (d, s)$ then $u_n(x) < \frac{(\lambda\tilde{a})^{\frac{1}{2-q}}}{2}$. By the maximality of (d, s) and the continuity of u_n we have that $u_n(d) = \frac{(\lambda\tilde{a})^{\frac{1}{2-q}}}{2}$, implying in particular that $d > -n$.

Since $u_n(x) < \frac{(\lambda\tilde{a})^{\frac{1}{2-q}}}{2}$ for all $x \in (d, s)$ and v_{k_i} converges uniformly to u_n, there exist $i_1 \in \mathbb{N}$ such that, for $i > i_1$ and $x \in (d, s)$,

$$v_{k_i}(x) < (\lambda\tilde{a})^{\frac{1}{2-q}}.$$

Then, by Lemma 32 $v''_{k_i}(x) < 0$. Using that $u_n(d) = \frac{(\lambda\tilde{a})^{\frac{1}{2-q}}}{2}$, there exist $i_2 \in \mathbb{N}$ such that $i > i_2$ implies

$$v_{k_i}(d) > \frac{(\lambda\tilde{a})^{\frac{1}{2-q}}}{4}.$$

Let $i_0 > \max\{i_1, i_2\}$ and define $f : (d,s) \to \mathbb{R}$ by

$$f(x) = \frac{(\lambda\tilde{a})^{\frac{1}{2-q}}}{4} \cdot \frac{x-s}{d-s}.$$

We have that $f(d) = \frac{(\lambda\tilde{a})^{\frac{1}{2-q}}}{4}$ and $f(s) = 0$. Let $U_i(x) = v_{k_i}(x) - f(x)$ for $i \geq i_0$, then

$$\begin{cases} U_i''(x) < 0, & \text{for } x \in (d,s) \\ U_i(d) > 0, U_i(s) = v_{k_i}(s) \geq 0. \end{cases} \tag{4.67}$$

By the maximum principle, the minimum of U_i is reached on the border of the interval (d,s), implying that $U_i(x) > 0$ for all $x \in (d,s)$, that is, $v_{k_i}(x) > f(x)$ for all $x \in (d,s)$ and $i \geq i_0$. Thus, taking $x = y$ and making $i \to \infty$, we obtain

$$u_n(y) \geq f(y) > 0,$$

contradiction.

Item 2. Since the estimation (4.54) holds, that is,

$$|v_k|_{1+\beta} \leq \hat{C}$$

for all $k \in \mathbb{N}$; and, for all $1 < \alpha < \beta$, we have compact embedding $C^{1,\beta}[-n,n] \hookrightarrow C^{1,\alpha}[-n,n]$, we may assume – taking a subsequence if necessary – that there exist $\widetilde{u_n} \in C^{1,\alpha}[-n,n]$ such that $v_k \to \widetilde{u_n}$ in $C^{1,\alpha}[-n,n]$ as $k \to \infty$. Thus

$$v_k \to u_n \text{ in } C^0[-n,n] \text{ as } k \to \infty$$
$$v_k \to \widetilde{u_n} \text{ in } C^{1,\alpha}[-n,n] \text{ as } k \to \infty.$$

Then for all $x \in [-n,n]$ we have

$$u_n(x) = \lim_{k\to\infty} v_k(x) = \widetilde{u_n}(x),$$

i.e, $u_n = \widetilde{u_n} \in C^{1,\alpha}[-n,n]$.

Considering the definition of weak solution, for all $\varphi \in H_0^1(-n,n)$

$$\int_{-n}^{n} A(v_k)v_k'\varphi' + \int_{-n}^{n} v_k\varphi = \int_{-n}^{n} (\lambda a_1|v_k|^{q-1} + |v_k|^{p-1})\varphi + \int_{-n}^{n} f_k(|v_k'|)\varphi + \int_{-n}^{n} \frac{\varphi}{k}.$$

By (D.C.T) is straightforward to see that the following convergences are true:

$$\int_{-n}^{n} A(v_k)v_k'\varphi' \to \int_{-n}^{n} A(u_n)u_n'\varphi',$$
$$\int_{-n}^{n} v_k\varphi \to \int_{-n}^{n} u_n\varphi,$$
$$\int_{-n}^{n} (\lambda a_1|v_k|^{q-1} + |v_k|^{p-1})\varphi \to \int_{-n}^{n} (\lambda a_1|u_n|^{q-1} + |u_n|^{p-1})\varphi,$$
$$\int_{-n}^{n} \frac{\varphi}{k} \to 0,$$

as $k \to \infty$. Let us examine the remaining integral. First notice that $f_k(|v_k'|)$ converges uniformly to $g(|u_n'|)$; indeed, given $\epsilon > 0$ there exists $k_0 \in \mathbb{N}$ such that $k > k_0$ implies

$$||v_k'(x)| - |u_n'(x)|| < \epsilon \quad \forall x \in [-n, n]. \tag{4.68}$$

Also there exist $0 < \delta_k < \epsilon$ such that if $|x - y| < \delta_k$, then

$$|f_k(x) - f_k(y)| < \frac{\epsilon}{2}; \tag{4.69}$$

thus for $k > k_0$

$$|f_k(|v_k'(x)|) - f_k(|u_n'(x)|)| < \frac{\epsilon}{2} \quad \forall x \in [-n, n]. \tag{4.70}$$

In the perspective of Theorem 18, f_k converges to g uniformly in bounded sets; since $\|u_n'\|_\infty \leq \hat{C}$, for $x \in [-\hat{C}, \hat{C}]$ there exist $k_1 \in \mathbb{N}$ such that $k > k_1$ implies

$$|f_k(x) - g(x)| < \frac{\epsilon}{2} \quad \forall x \in [-\hat{C}, \hat{C}] \tag{4.71}$$

and with all these ingredients we obtain the uniform convergence, because for $k > \max\{k_0, k_1\}$

$$|f_k(|v_k'(x)|) - g(|u_n'(x)|)| \leq |f_k(|v_k'(x)|) - f_k(|u_n'(x)|)| + |f_k(|u_n'(x)|) - g(|u_n'(x)|)| \tag{4.72}$$
$$< \epsilon \quad \forall x \in [-n, n]. \tag{4.73}$$

Thus, by (D.C.T)

$$\int_{-n}^{n} f_k(|v_k'|)\varphi \to \int_{-n}^{n} g(|u_n'|)\varphi$$

as $k \to \infty$. All these convergences together show that u_n is a weak solution for the problem (P_n).

From Theorem 4 from [18, Pag. 317, Chap. 6] we conclude that $u_n \in W^{2,2}(-n, n)$; and similarly to the argument showed in equation (4.55) we obtain that $u_n'' \in C^0(-n, n)$. Thus u_n is a strong solution to problem (P_n).

□

4.3 SOLUTION IN $\mathbb{R}$

To obtain a solution defined in $\mathbb{R}$ we will utilize a subsequence construction wrapping it up with the arguments presented in the last section. The reader should notice that the notation "u_n" used for the solution of (P_n) – previously obtained – in $(-n, n)$ is not accidental: extending u_n by zero out of $[-n, n]$ we obtain a sequence (u_n) in $H^1(\mathbb{R})$. Throughout this section we will use u_n to denote the solution "u_n" and its extension. Also one can see that $\|u_n\|_{H^1(\mathbb{R})} = \|u_n\|_{W^{1,2}(-n,n)} \leq r$ for all n.

Let $K_1 = [-1, 1]$; then for all $n \geq 1$ we have that $u_n^1 := u_n|_{K_1}$ is well defined and $u_n^1 \in H^1(-1, 1)$. By the limitation $\|u_n^1\|_{W^{1,2}(-1,1)} \leq r$ there exist a subsequence $u_{n,1}$ and $s_1 \in H^1(-1, 1)$ such that $u_{n,1} \rightharpoonup s_1$ in $H^1(-1, 1)$. Notice that the compact injection $H^1(-1, 1) \hookrightarrow C^0[-1, 1]$ implies that $u_{n,1} \to s_1$ in $C^0[-1, 1]$.

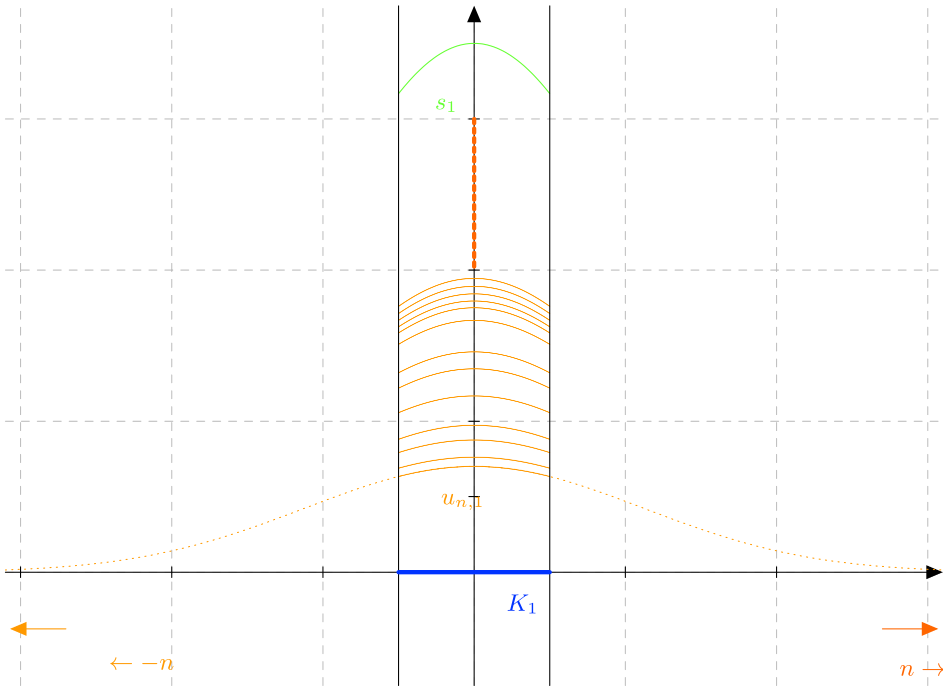

Figure 1 – Construction with K_1 to obtain s_1

Let $K_2 = [-2, 2]$; taking n in the set of indices of the subsequence $u_{n,1}$, for $n \geq 2$ we have that $u_n^2 := u_n|_{K_2}$ is well defined and $\|u_n^2\|_{W^{1,2}(-2,2)} \leq r$. Thus there exist a subsequence $u_{n,2}$ of u_n^2 and $s_2 \in H^1(-2, 2)$ such that $u_{n,2} \rightharpoonup s_2$ in $H^1(-2, 2)$.

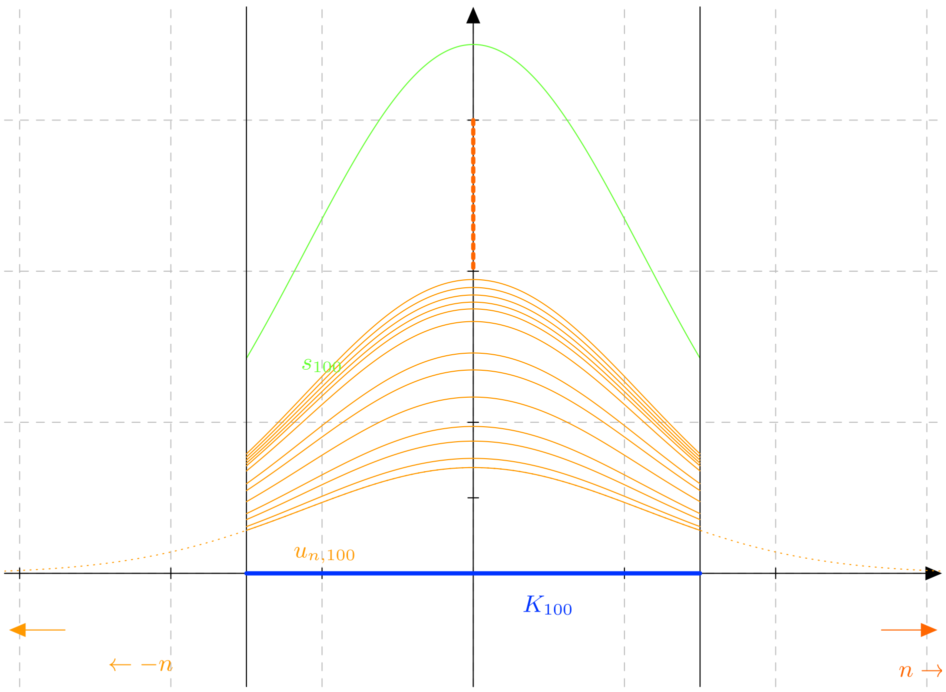

Figure 2 – Construction with K_{100} to obtain s_{100}

Repeating the same argument, by induction we get that for all $j \in \mathbb{N}$ there exist a subsequence $u_{n,j}$ of $u_{n,j-1}$ and $s_j \in H^1(-j,j)$ such that $u_{n,j} \rightharpoonup s_j$ in $H^1(-j,j)$.

Remark 15. The reader should read the notation "$u_{n,j}$" as follows: $u_{n,j}$ is the subsequence of u_n that converges weakly in $H^1(-j,j)$ to s_j. As mentioned this weak convergence implies convergence in $C^0[-j,j]$.

Lemma 33. $s_j|_{[1-j,j-1]} = s_{j-1}$

Proof. Given $x \in [1-j, j-1]$ we have that

$$s_{j-1}(x) = \lim_{n\to\infty} u_{n,j}(x) = s_j(x),$$

because $u_{n,j}$ is a subsequence of $u_{n,j-1}$. □

Define $v : \mathbb{R} \to \mathbb{R}$ by the following rule: given $x \in \mathbb{R}$ there exist a minimum $\tilde{t} \in \mathbb{N}$ such that $x \in [-\tilde{t}, \tilde{t}]$; then $v(x) = s_{\tilde{t}}(x)$. The previous lemma show us that v is well defined. This function v is the candidate of solution in $\mathbb{R}$, from here and forward we will focus our attention in proving that, in fact, v is a smooth non-zero solution.

Let T be a compact subset of $\mathbb{R}$ and $\tilde{t} \in \mathbb{N}$ such that $T \subset [-\tilde{t}, \tilde{t}]$. Defining $W : [-n,n] \times [-Cr, Cr] \times \mathbb{R} \longrightarrow \mathbb{R}$ by

$$W(x,z,p) = z - (\lambda a_1(x)|z|^{q-1} + |z|^{p-1} + g(|p|))$$

one can se that the estimation (4.53) also holds. Then, for any $n \geq \tilde{t}$, by Theorem 1 from [7] there exist $\hat{C}(\tilde{t}) > 0$ and $0 < \beta(\tilde{t}) \leq 1$ such that

$$|u_n|_{1+\beta(\tilde{t})} \leq \hat{C}(\tilde{t}) \text{ in } C^{1,\beta(\tilde{t})}[-\tilde{t}, \tilde{t}].$$

Taking $0 < \alpha(\tilde{t}) < \beta(\tilde{t}) \leq 1$ we get (see the argumentation on *Item 2* Proposition 31)

$$u_{n,\tilde{t}} \to v|_{[-\tilde{t},\tilde{t}]} \text{ in } C^{1,\alpha(\tilde{t})}[-\tilde{t}, \tilde{t}].$$

Thus, in particular, $v|_T \in C^1(T)$; since this is true for any T we conclude that $v \in C^1(\mathbb{R})$.

Now we will show that v is not identically null. Let $\widetilde{a}_{\tilde{t}} := \inf_{x\in[-\tilde{t},\tilde{t}]} a_1(x)$; we'll use the arguments presented in *Item 1* from Proposition 31 in the interval $[-\tilde{t}, \tilde{t}]$.

Case 1. There exist a subsequence $(u_{n_i})_{i\in\mathbb{N}}$ of $(u_{n,\tilde{t}})$ such that $u'_{n_i} \geq 0$ in $(-\tilde{t}, 0)$ for all i.

The analysis of this case follows exactly the same parameters of **Case 1** from *Item 1*, Proposition 31. The main difference is the change of $\tilde{a}$ to $\widetilde{a}_{\tilde{t}}$.

Case 2. For all subsequence of $(u_{n,\tilde{t}})$ there exist a subsubsequence $(u_{n_i})_{i\in\mathbb{N}}$ and exist a sequence $(z_i)_{i\in\mathbb{N}} \subset (-\tilde{t}, 0)$ such that $u'_{n_i}(z_i) < 0$.

For this case we can reformulate Lemma 32 as follows: If $x \in (-\tilde{t},\tilde{t})$ and $u''_{n_i}(x) \geq 0$, then $u_{n_i}(x) > (\lambda \tilde{a}_{\tilde{t}})^{\frac{1}{2-q}}$. This is true because we already know that u_{n_i} is strictly positive in $(-\tilde{t},\tilde{t})$, then the estimation

$$-A'(u_{n_i}(t))|u'_{n_i}(t)|^2 - A(u_{n_i}(t))u''_{n_i}(t) + u_{n_i}(t) > \lambda \tilde{a}_{\tilde{t}}|u_{n_i}(t)|^{q-1} \tag{4.74}$$

is immediately established. The remaining argumentation is similar.

Thus we conclude that $v|_T > 0$, then $v > 0$ in $\mathbb{R}$. At last, let $\varphi \in C_0^\infty(\mathbb{R})$ be such that $\text{supp}(\varphi) = T$. Then

$$\int_{-\tilde{t}}^{\tilde{t}} A(u_{n,\tilde{t}})u'_{n,\tilde{t}}\varphi' + \int_{-\tilde{t}}^{\tilde{t}} u_{n,\tilde{t}}\varphi = \int_{-\tilde{t}}^{\tilde{t}} (\lambda a_1|u_{n,\tilde{t}}|^{q-1} + |u_{n,\tilde{t}}|^{p-1})\varphi + \int_{-\tilde{t}}^{\tilde{t}} g(|u'_{n,\tilde{t}}|)\varphi.$$

When $n \to \infty$ we get

$$\int_{-\tilde{t}}^{\tilde{t}} A(v)v'\varphi' + \int_{-\tilde{t}}^{\tilde{t}} v\varphi = \int_{-\tilde{t}}^{\tilde{t}} (\lambda a_1|v|^{q-1} + |v|^{p-1})\varphi + \int_{-\tilde{t}}^{\tilde{t}} g(|v'|)\varphi.$$

Since T is any compact subset of $[-\tilde{t},\tilde{t}]$ we conclude that v is a weak solution of the problem

$$-(A(u)u')'(t) + u(t) = \lambda a_1(t)|u(t)|^{q-1} + |u(t)|^{p-1} + g(|u'(t)|) \tag{4.75}$$

in $(-\tilde{t},\tilde{t})$; by the Theorem 1 from [18, sect. 6.3, thm. 1] we have that $v \in H^2_{\text{loc}}(-\tilde{t},\tilde{t})$, thus – using the same arguments as in (4.55)– $v \in C^2(-\tilde{t},\tilde{t})$. Moreover, since there was no restriction over $\tilde{t}$, $v \in C^2(\mathbb{R})$ and is a solution for our main problem in $\mathbb{R}$. From Corollary 11 we get the homoclinic condition.

APPENDIX A – MISCELLANIES

This appendix contains certain proofs that were omitted through out this dissertation, for sake of clarity, from the main text of chapter 4.

Proposition 34. *Properties of* $\mathbb{E}_0^1(-n,n)$*:*

(I) it is a Hilbert space.

Proof. Indeed, it is clear that $\mathbb{E}_0^1(-n,n)$ is a normed vector space. Let (v_n) be a sequence in $\mathbb{E}_0^1(-n,n)$ such that $v_n \to v$ in $H_0^1(-n,n)$. For all n there exists a null measure subset of $(-n,n)$, say P_n, such that

$$v_n(t) = v_n(-t) \quad \forall t \in P_n^c.$$

Define $P = \cup_{n=1}^{\infty} P_n$. Due to the embedding $W^{1,2}(-n,n) \hookrightarrow C^0(-n,n)$ we get that $v_n \to v$ in $C^0(-n,n)$. Thus, making $n \to \infty$,

$$v(t) = v(-t) \quad \forall t \in P^c.$$

So $v \in \mathbb{E}_0^1(-n,n)$, showing that $\mathbb{E}_0^1(-n,n)$ is a closed subset of $H_0^1(-n,n)$. Thus $\mathbb{E}_0^1(-n,n)$ is a Hilbert space. □

(II) It is separable. (This follows from the fact that H_0^1 *is separable).*

(III) It has an orthonormal basis.

Proof. See [16]. □

Proposition 35. *The convergence stated in (4.26) is true, i.e,*

$$\int_{-n}^{n} \left(-v_M + \lambda |v_M|^{q-1} + |v_M|^{p-1} + f_k(|v_M'|) + \frac{\psi}{k} \right) (v_M - P_M v_0) \to 0 \textit{ as } M \to \infty.$$

Proof. Initially notice that, due to the orthogonal basis, $P_M v_0 \to v_0$ in $H_0^1(-n,n)$.

1. $\int_{-n}^{n} \frac{\psi}{k}(v_M - P_M v_0) \to 0.$

Indeed, $\int_{-n}^{n} \frac{\psi}{k}(v_M - P_M v_0) \leq \frac{1}{k}\|\psi\|_{L^2}\|v_M - P_M v_0\|_{L^2} \leq \frac{1}{k}\|\psi\|_{L^2} \underbrace{(\|v_M - v_0\|_{L^2} + \|v_0 - P_M v_0\|_{L^2})}_{\to 0}.$

2. $\int_{-n}^{n} f_k(v_M')(v_M - P_M v_0) \to 0.$

$$\begin{aligned} \left| \int_{-n}^{n} f_k(v_M')(v_M - P_M v_0) \right| &\leq \int_{-n}^{n} |f_k(v_M') - f_k(0)||v_M - P_M v_0| \\ &\leq c(k) \int_{-n}^{n} |v_M'||v_M - P_M v_0| \\ &\leq c(k)\|v_M'\|_{L^2}\|v_M - P_M v_0\|_{L^2} \\ &\leq c(k) r \underbrace{\|v_M - P_M v_0\|_{L^2}}_{\to 0}. \end{aligned}$$

3. $\int_{-n}^{n}(-v_M+\lambda a_1|v_M|^{q-1}+|v_M|^{p-1})(v_M-P_Mv_0)\to 0.$

$$\left|\int_{-n}^{n}(-v_M+\lambda a_1|v_M|^{q-1}+|v_M|^{p-1})(v_M-P_Mv_0)\right|\leq$$
$$\int_{-n}^{n}|v_M||v_M-P_Mv_0|+\int_{-n}^{n}\lambda|a_1||v_M|^{q-1}|v_M-P_Mv_0|+\int_{-n}^{n}|v_M|^{p-1}|v_M-P_Mv_0|$$
$$\leq \underbrace{Cr\|v_M-P_Mv_0\|_{L^2}+\lambda\tilde{a}(Cr)^{q-1}(2n)^{1/2}\|v_M-P_Mv_0\|_{L^2}+(Cr)^{p-1}(2n)^{1/2}\|v_M-P_Mv_0\|_{L^2}}_{\to 0}$$

□

Proposition 36. *The convergences of Lemma 28 are true.*

Proof. Let $\Phi_M,\Phi,\Psi_M,\zeta_M\in(\mathbb{E}_0^1(-n,n))^*$ be given by

$$\Phi_M(w)=\int_{-n}^{n}A(v_M)v_0'w'$$
$$\Phi(w)=\int_{-n}^{n}A(v_0)v_0'w'$$
$$\Psi_M(w)=\int_{-n}^{n}A(v_M)P_Mv_0'w'$$
$$\zeta_M(w)=\int_{-n}^{n}A(v_0)P_Mv_0'w'.$$

Let $w\in\mathbb{E}_0^1(-n,n)$;

(i) $|\Phi_M-\Phi|\to 0.$

$$\begin{aligned}|\Phi_M(w)-\Phi(w)|&=\left|\int_{-n}^{n}A(v_M)v_0'w'-A(v_0)v_0'w'\right|\\&\leq\left|\int_{-n}^{n}|A(v_M)-A(v_0)||v_0'w'|\right|\\&\leq\left(\int_{-n}^{n}|A(v_M)-A(v_0)|^2|v_0'|^2\right)^{1/2}\|w'\|_{L^2}.\end{aligned}$$

Notice that

$$\left(\int_{-n}^{n}|A(v_M)-A(v_0)|^2|v_0'|^2\right)^{1/2}\to 0 \text{ as } M\to\infty$$

because $A(v_M)\to A(v_0)$ pointwise, (since weak convergence in $H_0^1(-n,n)$ implies convergence in $C^0[-n,n]$), and there exists a constant $J>0$, (one can take $J>\max\{\tilde{A}r,A(r)\}$), such that

$$|A(v_M)-A(v_0)|^2|v_0'|^2\leq 4J^2|v_0'|^2.$$

Thus by (D.C.T) we obtain the desired convergence. Then we conclude that **(i)** holds.

(ii) $|\Psi_M-\Phi_M|\to 0.$

$$\begin{aligned}|\Psi_M(w)-\Phi_M(w)|&=\left|\int_{-n}^{n}A(v_M)w'(P_Mv_0'-v_0')\right|\\&\leq\int_{-n}^{n}|A(v_M)||w'||P_Mv_0'-v_0'|\\&\leq J\int_{-n}^{n}|w'||P_Mv_0'-v_0'|\\&\leq J\|w\|_{W^{1,2}}\|P_Mv_0'-v_0'\|_{L^2}.\end{aligned}$$

Since $\|P_M v_0' - v_0'\|_{L^2} \to 0$ as $M \to \infty$ we obtain **(ii)**.

(iii) $|\zeta_M - \Phi| \to 0$.

$$\begin{aligned} |\zeta_M(w) - \Phi(w)| &= \left| \int_{-n}^{n} A(v_0) w' (P_M v_0' - v_0') \right| \\ &\leq \int_{-n}^{n} |A(v_0)||w'||P_M v_0' - v_0'| \\ &\leq J \int_{-n}^{n} |w'||P_M v_0' - v_0'| \\ &\leq J \|w\|_{W^{1,2}} \|P_M v_0' - v_0'\|_{L^2}. \end{aligned}$$

Since $\|P_M v_0' - v_0'\|_{L^2} \to 0$ as $M \to \infty$ we obtain **(iii)**.

$\square$

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