UNIVERSIDADE FEDERAL DE JUIZ DE FORA

INSTITUTO DE CIÊNCIAS EXATAS

PROGRAM OF ACADEMIC MASTER'S DEGREE IN MATHEMATICS

Pedro de Oliveira Emerick

On (non)Lineability and (non)Spaceability in L_1 spaces

Juiz de Fora

2024

Pedro de Oliveira Emerick

On (non)Lineability and (non)Spaceability in L_1 spaces

Dissertation presented to the Graduate Program in Mathematics from Universidade Federal de Juiz de Fora, as required to obtain a Master's Degree in Mathematics. Area of concentration: Analysis

Advisor: Prof. Dr. Willian Versolati França

Juiz de Fora

2024

Ficha catalográfica elaborada através do Modelo Latex do CDC da UFJF
com os dados fornecidos pelo(a) autor(a)

Emerick, Pedro de Oliveira.
On (non)Lineability and (non)Spaceability in L_1 spaces / Pedro de Oliveira Emerick. – 2024.
44 f.

Advisor: Willian Versolati França
Dissertation (Master degree) – Universidade Federal de Juiz de Fora, Instituto de Ciências Exatas. Program of Academic Master's Degree in Mathematics, 2024.

1. Lineability. 2. Spaceability. 3. Sequence spaces. 4. L_p spaces I. França, Willian, orient. II. Título.

Pedro de Oliveira Emerick

On (non)Lineability and (non)Spaceability in L_1 spaces

Dissertação apresentada ao Programa de Pós-graduação em Matemática da Universidade Federal de Juiz de Fora como requisito parcial à obtenção do título de Mestre em Matemática. Área de concentração: Matemática Pura

Aprovada em 24 de julho de 2024.

BANCA EXAMINADORA

Prof Dr. Willian Versolati França - Orientador

Universidade Federal de Juiz de Fora

Prof Dr. Geraldo Márcio de Azevedo Botelho

Universidade Federal de Uberlândia

Prof Dr. Nelson Dantas Louza Júnior

Universidade Federal de Juiz de Fora

Juiz de Fora, 25/07/2024.

Documento assinado eletronicamente por **Willian Versolati Franca**, **Professor(a)**, em 25/07/2024, às 12:08, conforme horário oficial de Brasília, com fundamento no § 3º do art. 4º do Decreto nº 10.543, de 13 de novembro de 2020.

Documento assinado eletronicamente por **Nelson Dantas Louza Junior**, **Professor(a)**, em 25/07/2024, às 13:38, conforme horário oficial de Brasília, com fundamento no § 3º do art. 4º do Decreto nº 10.543, de 13 de novembro de 2020.

Documento assinado eletronicamente por **Geraldo Márcio de Azevedo Botelho**, **Usuário Externo**, em 25/07/2024, às 15:45, conforme horário oficial de Brasília, com fundamento no § 3º do art. 4º do Decreto nº 10.543, de 13 de novembro de 2020.

A autenticidade deste documento pode ser conferida no Portal do SEI-Ufjf (www2.ufjf.br/SEI) através do ícone Conferência de Documentos, informando o código verificador **1885009** e o código CRC **10EFD763**.

Dedico este trabalho a minha família, amigos e todos que me apoiaram.

ACKNOWLEDGEMENTS

Agradeço primeiramente a Deus por ter me dado forças e oportunidades para chegar até aqui.

Agradeço aos meus pais, Moisés e Sueli. O amor, apoio e encorajamento de vocês ao longo da minha vida foram fundamentais. O exemplo e os valores que vocês me transmitiram moldaram o caminho que percorri até alcançar esta conquista. Este trabalho também é de vocês.

Agradeço à Anna Júlia, minha noiva. Seu amor, apoio e incentivos constantes foram fundamentais para superar os momentos difíceis de minha jornada. Além disso, compartilhamos nos últimos 8 anos a paixão pela matemática. Foram incontáveis horas trabalhando juntos desde o ensino médio, passando pela graduação e agora no mestrado. Este trabalho é fruto de nossa parceria, na vida e nos estudos.

Agradeço a todos os professores que contribuíram para minha formação, tanto acadêmica quanto pessoal. Reservo um agradecimento especial a alguns que marcaram significativamente minha trajetória: ao professor Rodrigo, meu orientador no PIC Jr; aos professores Marcelo Salomão, Luiz Colatto e Fábio, do CEFET, que me iniciaram no mundo da matemática acadêmica; e à professora Cristiane, que me orientou durante toda a graduação.

Agradeço ao meu orientador, Willian. Os resultados contidos nessa dissertação são fruto de muitos meses de trabalho colaborativo. Agradeço em especial por ter me tratado não apenas como um aluno, mas como um parceiro de pesquisa. Graças a isso, pude crescer como matemático, ganhando maturidade e indepêndencia.

Agradeço também a todos meus amigos, colegas de curso e todos que dividiram comigo esse percurso. Agradeço à OBMEP e seus programas, que introduziram milhares de jovens à matemática, eu incluso. Agradeço à CAPES pelo financiamento. Um agradecimento especial ao Luan Arjuna, que trabalhou comigo por alguns meses enquanto escrevia esta dissertação.

Por fim, como não poderia faltar, agradeço a todos que não atrapalharam.

"Uma jornada de mil milhas começa com um único passo."
Lao Tzu.

RESUMO

Na presente dissertação, apresentaremos métodos para a construção de conjuntos que são lineáveis ou até mesmo espaçáveis em determinados espaços de Banach. Nossa abordagem também nos permitirá exibir exemplos de conjuntos que são lineáveis, mas não espaçáveis, e conjuntos que nem sequer são lineáveis. Seja $v = (v_n)$ um elemento de ℓ_1 com apenas um número finito de entradas nulas. Neste cenário discutiremos a lineabilidade (espaçabilidade) dos seguintes conjuntos: $B(v)$ (respectivamente $A_0(v)$) o conjunto de todos os elementos de ℓ_1 onde o teste da comparação por limite com relação a v é conclusivo (respectivamente inconclusivo); $X(v)$ o conjunto de todos os elementos de ℓ_1 onde o teste da comparação padrão falha (com relação a v). Neste contexto provaremos que o conjunto $A_0(v)$ é $\mathfrak{c}$-denso-lineável mas não é espaçável. Por outro lado, o conjunto $B(v) \cup \{0\}$ contém apenas subespaços de dimensão 1. Além disso, nossos métodos nos permitirão concluir: (1) todo subespaço fechado de dimensão infinita de ℓ_1 contém um elemento de $X(v)$; (2) $X(v)$ é $\mathfrak{c}$-denso-lineável e $\mathfrak{c}$-espaçável. Utilizando os resultados supracitados, provaremos que o conjunto formado pelos elementos de ℓ_1 cujo teste da raiz (respectivamente razão) é inconclusivo é de fato espaçável. Também provaremos alguns resultados clássicos. Por exemplo, concluiremos que todo subespaço fechado de dimensão infinita de ℓ_1 contém um elemento com infinitas entradas nulas. Ao final estenderemos alguns desses resultados para o caso $L_1(\mathscr{M})$, onde $\mathscr{M}$ é um conjunto ilimitado de um espaço vetorial normado fixo Y, e $\mathscr{M}$ está munido com a σ-álgebra de Borel.

Palavras-chave: Lineabilidade. Espaçabilidade. Espaços de sequências. Espaços L_p.

ABSTRACT

In the present dissertation, we will provide methods for constructing lineable or even spaceable sets in certain Banach spaces. Our approach will also allow us to exhibit examples of sets that are lineable but not spaceable, and sets that are not even lineable. Let $v = (v_n)$ be an element of ℓ_1 with finitely many zero entries. In this setting, we will discuss the lineability (spaceability) of the following sets: $B(v)$ (resp. $A_0(v)$) the set of all elements of ℓ_1 where the limit comparison test with respect to v works (resp. fails); $X(v)$ the set of all elements of ℓ_1 where the standard comparison test fails (with respect to v). On this matter, we will prove that the set $A_0(v)$ is $\mathfrak{c}$-dense-lineable but not spaceable. Meanwhile, the set $B(v) \cup \{0\}$ only contains finite-dimensional subspaces of dimension 1. Moreover, our methods will allow us to conclude: (1) every infinite-dimensional closed subspace of ℓ_1 contains an element of $X(v)$; (2) $X(v)$ is $\mathfrak{c}$-dense-lineable and $\mathfrak{c}$-spaceable. As an application of our above mentioned findings, we will be able to conclude that the set formed by all elements of ℓ_1 for whose generated series the root (resp. ratio) test fails is spaceable. In addition, we will also retrieve some known results. For instance, we will prove that every infinite-dimensional closed subspace of ℓ_1 contains an element with infinitely many zeros. At the end, we will extend some of these results to the case $L_1(\mathscr{M})$, where $\mathscr{M}$ is an unbounded subset of a fixed normed vector space Y, and $\mathscr{M}$ is equipped with the Borel σ-algebra.

Keywords: Lineability. Spaceability. Sequence spaces. L_p spaces.

SUMMARY

1 Introduction

In the last decades, the topic of Lineability has captivated the attention of many researchers worldwide. This line of investigation deals with the problem of deciding whether or not a given subset M (of some vector space X) with some sort of pathological behavior or enjoying a certain special property contains (up to the zero vector in some cases) an infinite-dimensional (closed) subspace. Problems of this nature have their roots on the seminal work of Gurariy [1] who proved that the set of continuous nowhere differentiable functions on the interval $[0,1]$ contains an infinite-dimensional subspace, in other words, he showed that this set is lineable. In 1999, Fonf et.al [2] proved that this set is actually spaceable, that is, it contains an infinite-dimensional closed subspace.

Since Gurariy's work [1] it has been an enormous number of published results on this topic. For instance, the reader may see either the papers [3, 4] where the authors considered certain functions spaces or the references [3, 5, 6, 7, 8, 9] in the context of sequence spaces just to mention a few. We also recommend to the interested reader to see the book [10] for a more complete exposition about lineability and some other related topics (for instance algebrability), and the papers [11, 12, 13] for a more modern treatment on this subject. In the present thesis, we will proceed in this line of investigation by presenting new results on this topic. Our work is organized as follows.

In the second chapter, we give a short exposition regarding the cardinality of sets which turns out to be a standard prerequisite for many results in this field. Moreover, in order to make the work self-contained, we formally introduce the concepts of lineability, spaceability and some other related definitions.

Chapter 3 is the bulk of our work. Here, we discuss the (non)lineability (spaceability) of certain subsets of ℓ_1. Before outlining our results we are going to establish some notation. For an arbitrary element $x = (x_n) \in \ell_1$, the sets $E(x) = \{n \in \mathbb{N} \mid x_n \neq 0\}$ and $D(x) = (\mathbb{N} \setminus E(x))$ are the support and the kernel of x respectively. Hereafter, we assume that $v = (v_n)$ and d are two fixed elements of ℓ_1 and $\overline{\mathbb{R}}$, respectively, where $D(v)$ is finite and $0 \leq d \leq \infty$. We will adopt the convention that every limit or supremum of the form (w_n/v_n) is taken over $E(v)$. That being said, we will simply write

$$\lim_{\substack{n\to\infty \\ n\in E(v)}} (w_n/v_n) = \lim_{n\to\infty} (w_n/v_n) \text{ and } \sup_{n\in E(v)} (w_n/v_n) = \sup_{n\in\mathbb{N}} (w_n/v_n).$$

The sets that will be considered in chapter 3 are the following ones: M, R, $N = (\ell_1 \setminus M)$, $S = (\ell_1 \setminus R)$, $A_d(v)$, and $X(v)$ where

$$M = \left\{ a = (a_n) \in \ell_1 \mid \lim_{n\to\infty} \sup |a_n|^{1/n} = 1 \right\},$$

$$R = \left\{ a = (a_n) \in \ell_1 \middle| \lim_{n\to\infty} \sup \left| \frac{a_{n+1}}{a_n} \right| \geq 1 \right\},$$

$$A_d(v) = \left\{ w = (w_n) \in \ell_1 \mid \lim_{n\to\infty} \left| \frac{w_n}{v_n} \right| = d \text{ and } D(w) \text{ is finite} \right\},$$

and

$$X(v) = \left\{ w = (w_n) \in \ell_1 \mid \sup_{n\in\mathbb{N}} \left| \frac{w_n}{v_n} \right| = \infty \right\}.$$

Among other results, we prove that:

- $A_0(v)$, $A_\infty(v)$ and $X(v)$ are maximal dense-lineable (Theorems 3.2.3 and 3.2.4).
- $A_0(v)$ and $A_\infty(v)$ are not spaceable (Theorems 3.4.1 and 3.4.2).
- $X(v)$ is spaceable (Theorem 3.3.3).
- $B(v) = \cup_{0<d<\infty} A_d(v)$ is only 1-lineable (Theorem 3.2.1).
- $A_0(v), A_\infty(v)$ are meager sets and $X(v)$ is residual (Theorems 3.2.5 and 3.2.6).

Applying the results above, we obtain the following (Theorems 3.4.3 and 3.4.4):

- M and R are maximal dense-lineable.
- N and S are maximal dense-lineable.
- M and R are spaceable.
- N and S are not spaceable.
- M and R are residual.
- N and S are meager.

The methods employed in this chapter allowed us to recover and generalize some of the results of G. Araújo et. al. [3, Theorem 6.2]. We also want to mention that results involving either nonspaceability or nonlineability are quite scarce in the literature.

In Chapter 4 we replace the set ℓ_1 with $L_1(\mathscr{M})$, where $\mathscr{M}$ is an unbounded set of a normed vector space Y, where $\mathscr{M}$ is endowed with the Borel σ-algebra, and we generalize some of the results that can be found in Chapter 3. To this end, we introduce the notion of *convergence in measure at infinity* in $L_1(\mathscr{M})$ which can be seen as a suitable generalization of the notion of limit at infinity for the space $\ell_1 = L_1(\mathbb{N})$ provided that $\mathbb{N}$ is equipped with the counting measure on the power set $\mathscr{P}(\mathbb{N})$. This notion of convergence enabled us to define the sets $A_0(f)$, $A_\infty(f)$, and $B(f)$ in the same fashion as the ones in Chapter 3 (provided that such limit is unique - a feature that can not be taken for granted in general). In this setting, we were able to prove that $A_0(f)$, $A_\infty(f)$ are $\mathfrak{c}$-lineable and $B(f)$ is only 1-lineable. We also present necessary and sufficient conditions to decide when: (1) the space $L_1(\mathscr{M})$ is infinite-dimensional; (2) to ensure that the limit in measure at infinity is unique on $L_1(\mathscr{M})$.

Throughout this work, we do expect the reader to have some basic knowledge of functional analysis, measure theory and topology. Some background in elementary set theory is also desirable including the notions of cardinal numbers and its arithmetic, since they can come to be useful for some lineability problems. However, in the present work, we are not going to make use of any elaborate set-theoretic argument. It may sound as a surprise, but we do not require the reader to have any previous experience with the theory of lineability.

2 Basic definitions and results

This chapter is dedicated to presenting the basic definitions and terminology to be used as the starting point of our work. At first we will give a brief exposition regarding cardinal numbers by formally introducing some elementary notions and stating some known results. In the second part we will proceed on a similar way with the concept of lineability instead.

2.1 Cardinality

Cardinal numbers and cardinal arithmetic are two necessary tools when it comes to some lineability problems, and for this reason we have decided to cover the basics on this topic. This subsection is based on the reference [10] - sections I and II - which is a recommended source for a broader exposition on the prerequisites for lineability regarding cardinal numbers. The upcoming proofs will be omitted since they are not necessary for our purposes and cardinality is not the main focus of the present dissertation.

Definition 2.1.1. Denoting by $\text{card}(A)$ the cardinality of a set A, then for non-empty sets A and B we say:

- $\text{card}(A) \leq \text{card}(B)$ if there is an injection from A to B.
- $\text{card}(A) \geq \text{card}(B)$ if there is a surjection from A to B.
- $\text{card}(A) = \text{card}(B)$ if there is a bijection from A to B.
- $\text{card}(A) < \text{card}(B)$ if $\text{card}(A) \leq \text{card}(B)$ and there is no bijection from A to B.
- $\text{card}(A) > \text{card}(B)$ if $\text{card}(A) \geq \text{card}(B)$ and there is no bijection from A to B.

Intuitively we may expect that the relation $\leq$ defined above behaves at some level as the usual order on $\mathbb{N}$. To begin with, it is quite clear that if $\text{card}(A) \leq \text{card}(B)$ and $\text{card}(B) \leq \text{card}(C)$, then $\text{card}(A) \leq \text{card}(C)$ (just compose the injection from A to B with the injection from B to C). The next two results also yield what one would usually expect.

Proposition 2.1.1. $\text{card}(A) \leq \text{card}(B)$ *if and only if* $\text{card}(B) \geq \text{card}(A)$.

Proof. See [10, Proposition I.2]. □

Theorem 2.1.1 (Cantor-Bernstein-Schröeder)**.** *If* $\text{card}(A) \leq \text{card}(B)$ *and* $\text{card}(A) \geq \text{card}(B)$*, then* $\text{card}(A) = \text{card}(B)$.

Proof. [10, Theorem I.3]. □

We define cardinal number (or simply cardinal) as card(A) for some set A. We say that a cardinal α is an infinite cardinal when $\alpha = \text{card}(A)$ for some infinite set A. If two sets A and B possess $\text{card}(A) = \text{card}(B)$ we may say that A and B belong to the same equivalence class - be careful since the collection of all sets does not constitute a set in the Z.F.C set theory. In the case that A is a finite set there exists $n \in \mathbb{N}$ such that $\text{card}(A) = \text{card}(A_n)$, where $A_n = \{1, 2, \cdots, n\}$. So, we may write $\text{card}(A) = n$. It is also convenient to define $\text{card}(\varnothing) = 0$. On this matter, we write $\text{card}(\mathbb{N}) = \aleph_0$ (read as "aleph zero") and $\text{card}(\mathbb{R}) = \mathfrak{c}$ (the cardinality of the continuum).

For simplicity, we usually denote cardinal numbers by Greek letters. The next result establishes that any two cardinals can be compared. The proof of this fact uses the famous Zorn's Lemma or one of the equivalent forms like the Axiom of Choice - for any given family of nonempty sets, their cartesian product is also a nonempty set. The latter is also equivalent to the existence of a basis for vector spaces.

Theorem 2.1.2. *If α, β are cardinals, then $\alpha \leq \beta$ or $\alpha \geq \beta$.*

Proof. [10, Theorem I.5]. □

Remark 2.1.1. If A is a set, then $\text{card}(\mathscr{P}(A)) > \text{card}(A) = \alpha$, where $\mathscr{P}(A)$ is the power set of A. In addition, if α is an infinite cardinal, then $\alpha \geq \aleph_0$.

When dealing with cardinal numbers, we may see ourselves intrigued on knowing the cardinality of a union of two sets or even the cardinality of a difference of sets. The notion of cardinal arithmetic allows us to make these computations. Now, let us define the operations between cardinals.

Definition 2.1.2. Let α, β be cardinal numbers. We set:

- $\alpha + \beta := \text{card}(S)$, where $S = A \cup B$, with $\alpha = \text{card}(A)$, $\beta = \text{card}(B)$ and $A \cap B = \varnothing$.
- $\alpha\beta := \text{card}(P)$, where $P = A \times B$, with $\alpha = \text{card}(A)$ and $\beta = \text{card}(B)$.
- $\alpha^\beta := \text{card}(C)$, where C is any set of the form $C = \prod_{i \in I} A_i$, with $\text{card}(I) = \beta$ and $\text{card}(A_i) = \alpha$ for all $i \in I$. Equivalently, if $\text{card}(A) = \alpha$, then $\alpha^\beta = \text{card}(A^I)$, with $A^I = \{f \mid f \text{ is a function from } I \text{ to } A\}$.

Clearly the notions defined above are independent of the choice of the sets A and B. In addition, for all cardinal numbers $\alpha, \beta, \gamma \geq 1$ one may verify that the following properties hold:

- $\alpha + \beta = \beta + \alpha$.
- $(\alpha + \beta) + \gamma = \alpha + (\beta + \gamma)$.

- $\alpha\beta = \beta\alpha$.
- $(\alpha\beta)\gamma = \alpha(\beta\gamma)$;
- $\alpha(\beta + \gamma) = \alpha\beta + \alpha\gamma$.
- $(\alpha\beta)^\gamma = \alpha^\gamma\beta^\gamma$.
- $\alpha^{\beta+\gamma} = \alpha^\beta\alpha^\gamma$.
- $(\alpha^\beta)^\gamma = \alpha^{\beta\gamma} = (\alpha^\gamma)^\beta$.

Remark 2.1.2. The relation $\leq$ is compatible with the operations between cardinals in the following sense: If $\alpha_1 \leq \alpha_2$ and $\beta_1 \leq \beta_2$, then $\alpha_1 + \beta_1 \leq \alpha_2 + \beta_2$, $\alpha_1\beta_1 \leq \alpha_2\beta_2$ and $\alpha_1^{\beta_1} \leq \alpha_2^{\beta_2}$.

We end this section with a collection of useful results and observations.

Theorem 2.1.3. *Let α, β be cardinal numbers with $1 \leq \beta \leq \alpha$ and α infinite. Then $\alpha + \beta = \alpha$.*

Proof. [10, Theorem II.2]. □

Corollary 2.1.1. *If α, β, γ are infinite cardinal numbers with $\alpha + \beta = \gamma$ and $\alpha < \gamma$, then $\beta = \gamma$.*

Remark 2.1.3. The corollary above says that, if A, B, C are infinite sets, and $A = B \cup C$ where $\mathrm{card}(B) < \mathrm{card}(A)$, then $\mathrm{card}(A) = \mathrm{card}(C)$. This fact has plenty of applications in lineability.

Theorem 2.1.4. *Let α, β be cardinal numbers, with $1 \leq \beta \leq \alpha$ and α infinite. Then $\alpha\beta = \alpha$.*

Proof. [10, Theorem II.4]. □

Corollary 2.1.2. *If β is an infinite cardinal, then $\aleph_0\beta = \beta$.*

The following three results are also of common use.

Proposition 2.1.2. $2^{\aleph_0} = \mathfrak{c}$.

Proof. [10, Proposition II.6]. □

Proposition 2.1.3. $\mathfrak{c}^{\aleph_0} = \mathfrak{c}$.

Proof. [10, Proposition II.8]. □

Remark 2.1.4. From the last result it is also possible to prove that if A is a set with $\mathrm{card}(A) = \mathfrak{c}$, then the set of sequences of elements of A has cardinality $\mathfrak{c}$. In particular, the set of all real or complex sequences has cardinality $\mathfrak{c}$.

2.2 Lineability, dense-lineability and spaceability

We start this subsection by recalling some definitions and results in the context of vector spaces.

Definition 2.2.1. A topological vector space $X = (X, \tau)$ is a vector space X over the field $\mathbb{K}$ ($\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$) endowed with a topology τ in a way that the scalar multiplication and the sum of vectors are continuous operations. Here, we understand tacitly that $\mathbb{K}$ is equipped with the usual topology.

Definition 2.2.2. Let X be a vector space over a field $\mathbb{K}$. A subset $\mathscr{B} \subset X$ is called a Hamel base (or simply a basis), if $\mathscr{B}$ is linearly independent and $\operatorname{span}(\mathscr{B}) = X$.

Remark 2.2.1. As a consequence of Zorn's Lemma we know that every vector space contains a basis. See [14, 4.1-7].

The following interesting result concerning basis holds:

Theorem 2.2.1. *Any two Hamel basis of a vector space X have the same cardinality.*

Proof. [10, Theorem III.4]. □

Due to the last theorem, for a given vector space X over a field $\mathbb{K}$, we may define $\dim(X) = \dim_{\mathbb{K}}(X)$ as the cardinality of any Hamel basis of X. We then have the following result:

Proposition 2.2.1. *Let X be a vector space over $\mathbb{R}$ or $\mathbb{C}$. Then* $\operatorname{card}(X) = \mathfrak{c}\dim(X) = \max\{\mathfrak{c}, \dim(X)\}$.

Proof. [15, Lemma 9.5.1]. □

In the case that X is a real or complex Banach space, the following proposition establishes a lower bound for $\dim(X)$ in the case that $\dim(X)$ is infinite.

Proposition 2.2.2. *If X is an infinite-dimensional Banach space, then* $\dim(X) \geq \mathfrak{c}$.

Proof. [10, Proposition III.5]. □

Now, we are in position to introduce the definition of lineability, spaceability and their variations.

Definition 2.2.3. Let X be a topological vector space, let M be a subset of X and let α be a cardinal number.

- M is said to be α-lineable if $M \cup \{0\}$ contains a vector space of dimension α. We say that M is maximal lineable when $\alpha = \dim(X)$.

- M is said to be α-spaceable if $M \cup \{0\}$ contains a closed vector space of dimension α. We say that M is maximal spaceable when $\alpha = \dim(X)$.

- M is said to be α-dense-lineable if $M \cup \{0\}$ contains a dense vector space of dimension α. We say that M is maximal dense-lineable when $\alpha = \dim(X)$.

It is standard to call the set M as lineable, spaceable, or dense-lineable when $\alpha \geq \aleph_0$ on each corresponding definition. We also want to point out that the definition concerning lineability remains valid without a topology on X.

The next example shows that there may not exist a maximum cardinal α such that M is α-lineable.

Example 2.2.1. Let X be an infinite-dimensional vector space and let $\{e_n \mid n \in \mathbb{N}\}$ be a linearly independent set. Let $j_1 < k_1 < j_2 < k_2 < j_3 < \cdots < k_m < j_{m+1} < \cdots$ be natural numbers with $k_m - j_m = m$ for each $m \in \mathbb{N}$, and let $M := \bigcup_{m=1}^{\infty} \operatorname{span}\{e_i \mid j_m \leq i \leq k_m\}$. By construction, the set M is n-lineable for all $n \in \mathbb{N}$. Now, we prove that M is not $\aleph_0$-lineable. If Y is a linear subspace contained in M, then $Y \subset \operatorname{span}\{e_i \mid j_m \leq i \leq k_m\}$ for some $m \in \mathbb{N}$ (that is, Y is finite-dimensional). Indeed, let us suppose that $x \in \operatorname{span}\{e_i \mid j_m \leq i \leq k_m\} \cap Y$ and $y \in \operatorname{span}\{e_i \mid j_l \leq i \leq k_l\} \cap Y$ with $m \neq l$ and x, y nonzero. Since $(x + y) \in Y \subset M$ then $(x + y) \in \operatorname{span}\{e_i \mid j_r \leq i \leq k_r\}$ for some $r \in \mathbb{N}$. Therefore, $x + y = z$ for some $z \in \operatorname{span}\{e_i \mid j_r \leq i \leq k_r\}$. Hence, $x + y - z = 0$ is a nontrivial linear combination of $\{e_n \mid n \in \mathbb{N}\}$ resulting in 0, which is a contradiction.

Remark 2.2.2. Here we are going to remark on some easy-to-check and illustrative facts ($M \subset X$).

- If M is not a dense subset of X, then M cannot be dense-lineable. Moreover, if X is nonseparable and M is at most $\aleph_0$-lineable, then M is not dense-lineable. More generally, if the least cardinality of a dense subset of X is $d(X)$, then a necessary condition for M to be dense-lineable is that M is $d(X)$-lineable.

- If X is a Banach space and M is not $\mathfrak{c}$-lineable, then M is not spaceable, since any infinite-dimensional Banach space has dimension at least $\mathfrak{c}$. Conversely, every spaceable subset M of a Banach space is at least $\mathfrak{c}$-lineable.

- We should not fool ourselves by thinking that the maximal lineability of a given set M implies that $X \setminus M$ is too small to be lineable. For instance, let X be a vector space with basis $\{e_n \mid n \in \mathbb{N}\}$ and $\mathbb{N} = \bigcup_{i=1}^{\infty} N_i$ where each N_i is infinite and $N_i \cap N_j = \varnothing$ for $i \neq j$. We define $X_i := \operatorname{span}\{e_i \mid i \in N_i\}$ for each $i \in \mathbb{N}$. Then, each X_i is maximal lineable and $\bigcup_{j \neq i} X_j \subset (X \setminus X_i) \cup \{0\}$. In sumary, X_i is maximal lineable, meanwhile there are infinitely many linearly independent subspaces of maximal dimension in $(X \setminus X_i) \cup \{0\}$.

In order to illustrate the theory, we close this section with a relatively simple construction that can be found in [7, Theorem 2.1]. For our purposes, let $\mathbb{K}$ to be either $\mathbb{C}$ or $\mathbb{R}$. The symbol $CS(\mathbb{K})$ represents the vector space (over $\mathbb{K}$) formed by all scalar sequences in $\mathbb{K}$ whose associated series converges. We say that a series is conditionally convergent when $\sum_{n=1}^{\infty} |x_n| = \infty$ but $(x_n) \in CS(\mathbb{K})$. Let M be the set of all sequences of $CS(\mathbb{K})$ whose associated series is conditionally convergent. Observe that M is not a vector space. We will prove that M is $\mathfrak{c}$-lineable. Before providing such proof, we need the following technical lemma.

Lemma 2.2.1. *There exists a family $\{A_\alpha \mid \alpha \in I\}$ of infinite subsets of $\mathbb{N}$ such that $\operatorname{card}(I) = \mathfrak{c}$ and $A_\alpha \cap A_\beta$ is a finite set whenever $\alpha, \beta \in I$ and $\alpha \neq \beta$.*

Proof. Let $\{q_n \mid n \in \mathbb{N}\}$ be the set $\mathbb{Q} \cap [0,1]$, and let I be the set of all irrationals numbers on the interval $[0,1]$. Since $\operatorname{card}(\mathbb{Q} \cap [0,1]) = \aleph_0$, then $\operatorname{card}(I) = \mathfrak{c}$. From the density of $\mathbb{Q} \cap [0,1]$ in $[0,1]$, for each $\alpha \in I$, we may choose a subsequence (q_{n_k}) of $\{q_n \mid n \in \mathbb{N}\}$ fulfilling the condition $\lim_{k\to\infty} q_{n_k} = \alpha$. So, we may define $A_\alpha = \{n_k \mid k \in \mathbb{N}\}$. By construction, each A_α is infinite and $A_\alpha \cap A_\beta$ is always finite whenever $\alpha \neq \beta$. $\square$

A family $\{A_\alpha \mid \alpha \in I\}$ as described in the last lemma is called almost disjoint. Such families are used in numerous proofs involving lineability problems, especially the ones considered in sequence spaces.

Theorem 2.2.2. *The set M is $\mathfrak{c}$-lineable in $CS(\mathbb{K})$.*

Proof. Let us fix any conditionally convergent series $\sum_{i=1}^{\infty} a_i$ such that $a_i \neq 0$ for all $i \in \mathbb{N}$. Let $\{A_\alpha \mid \alpha \in I\}$ be an almost disjoint family as described in Lemma 2.2.1. Now, fix $\alpha \in I$ and consider $A_\alpha = \{m_1 < m_2 < m_3 < \ldots < m_k < m_{k+1} < \ldots\}$. Thus, we define $x^{(\alpha)} = (x_n^{(\alpha)_{n\in\mathbb{N}}})$ given by $x_n^{(\alpha)} = a_k$ if $n = m_k$ and 0 otherwise. Note that $x^{(\alpha)} \in M$ for each $\alpha \in I$. Let $E = \operatorname{span}\{x^{(\alpha)} \mid \alpha \in I\}$. Now, let $\lambda_1, \cdots, \lambda_n$ be nonzero scalars and $\{\alpha_1, \cdots, \alpha_n\} \subset I$. We will prove that $z = \lambda_1 x^{(\alpha_1)} + \cdots + \lambda_n x^{(\alpha_n)} \in M$. Indeed, since $\{A_\alpha \mid \alpha \in I\}$ is almost disjoint, we may find a set $A \subset A_{\alpha_1}$ such that $A_{\alpha_1} \setminus A$ is finite and $A \cap (A_{\alpha_2} \cup \cdots \cup A_{\alpha_n}) = \varnothing$. Then $\sum_{i\in A} z_i = \sum_{i\in A} \lambda_1 x_i^{(\alpha_1)}$ is conditionally convergent. Hence, $z \in M$. The latter implies $E \subset M \cup \{0\}$. In particular, since $z \neq 0$ we may derive the linear independence of $\{x^{(\alpha)} \mid \alpha \in I\}$. Consequently M is $\mathfrak{c}$-lineable. $\square$

3 Lineability in sequence spaces

In the present chapter, we understand that the reader is familiar with all the standard notations related to the classical sequence spaces. In addition, we refer the reader to the introduction to recall the definition of the sets that will be our main object of investigation in this chapter, namely, M, R, $N = (\ell_1 \setminus M)$, $S = (\ell_1 \setminus R)$, $A_d(v)$, and $X(v)$ (for $0 \leq d \leq \infty$ and $v = (v_n) \in \ell_1$ with finitely many nonzero entries).

3.1 An overview of the main conclusions and a preliminary result

Let M, R, $N = (\ell_1 \setminus M)$, $S = (\ell_1 \setminus R)$, $A_d(v)$, and $X(v)$ as in the above paragraph. Now, let us exploit the meaning of the above mentioned sets. To begin with, the set M (resp. R) can be viewed as the set of sequences in ℓ_1 for whose generated series the root (resp. ratio) test fails. In the same fashion, the remaining sets can be seen - with respect to the element v - as: $B(v) = \cup_{0<d<\infty} A_d(v)$ is the set of all elements of ℓ_1 for which the limit comparison test works; $X(v)$ is the set where the standard comparison test fails - note that $A_\infty(v) \subset X(v)$. Note that the set $A_0(v)$ may be tricky, once it only contains the elements $w = (w_n)$ where the limit comparison test fails and $D(w) = \{n \in \mathbb{N} \mid w_n \neq 0\}$ is finite.

In [3, Theorem 6.2] G. Araújo et. al. showed that the sets M and R are maximal dense-lineable - meaning that M (resp. R) contains an infinite-dimensional dense subspace whose dimension is exactly equal to $\dim(\ell_1) = \mathfrak{c}$. Motivated by the latter result, we decided to investigate the (maximal) lineability (spaceability) of the sets $B(v)$, $X(v)$ and $A_0(v)$. The reader may now ask a somewhat natural question: Are the sets $A_d(v)$ (resp. $X(v)$) non-empty? For $d \in (0, \infty)$, it is quite clear that $dv \in A_d(v)$. The next proposition guarantees the non-voidness of $A_\infty(v)$.

Proposition 3.1.1. *The set $A_\infty(v)$ is non-empty. In particular, $X(v)$ is non-empty.*

Proof. Let construct an element $w = (w_n) \in A_\infty(v)$ as follows. First, for each $k \in \mathbb{N}$, we choose $n_k \in \mathbb{N}$ satisfying $\sum_{n=n_k}^{\infty} |v_n| < 1/2^{2k}$. Besides, we may assume without loss of generality that $n_{k-1} < n_k$ for each $k \in \mathbb{N}$. Second, we set $w_n = 2^k v_n$ if $n_k \leq n < n_{k+1}$, and $w_n = v_n$ otherwise. By construction, we have $D(w)$ finite and $|w_n/v_n| = 2^k$ whenever $n_k \leq n < n_{k+1}$. Therefore, $\lim_{n\to\infty} |w_n/v_n| = \infty$. Now, the proof boils down to showing that that $(w_n) \in \ell_1$. Indeed, on the one hand, we have

$$\sum_{n=1}^{\infty} |w_n| = \sum_{n=1}^{n_1-1} |w_n| + \sum_{k=1}^{\infty} \sum_{n=n_k}^{n_{k+1}-1} |w_n|.$$

On the other hand,

$$\sum_{n=n_k}^{n_{k+1}-1} |w_n| = 2^k \left[\sum_{n=n_k}^{n_{k+1}-1} |v_n| \right] < 2^k \left[\sum_{n=n_k}^{\infty} |v_n| \right] < 1/2^k.$$

Therefore, $(w_n) \in \ell_1$. For the second part, it suffices to note that $A_\infty(v) \subset X(v)$. □

For expository reasons, the proof of the fact that $A_0(v)$ is also non-empty will be presented in the next section (Proposition 3.2.1). Now, we are finally in the position to outline our main results. The chapter is organized as follows.

In Section 3.2, we will prove that the following two sets are $\mathfrak{c}$-dense-lineable: $A_\infty(v)$ (Theorem 3.2.3) and $A_0(v)$ (Theorem 3.2.4). In particular, $B_0(v) = \cup_{0\le d<\infty} A_d(v)$ is $\mathfrak{c}$-dense-lineable. However, we will see that the set $B(v) = \cup_{0<d<\infty} A_d(v)$ only contains finite-dimensional subspaces of dimension 1 (Theorem 3.2.1). At the end we will show that the sets $A_\infty(v)$, $B(v)$ and $B_0(v)$ are all meager sets.

In Section 3.3, we will show that every infinite-dimensional closed subspace of ℓ_1 contains an element of $X(v)$ (Theorem 3.3.2). Besides, we will see that $X(v)$ is $\mathfrak{c}$-spaceable (Theorem 3.3.3). Employing the same methods of Theorem 3.3.2, we will be able to show that every infinite-dimensional closed subspace of ℓ_1 contains a nonzero element with infinitely many zero entries (Theorem 3.3.1) - this last result was first proved in [5, Corollary 3.4].

Section 3.4 is dedicated to applications of the results obtained in Section 3.3. More precisely, we will verify that the sets $A_\infty(v)$ and $B_0(v)$ are not spaceable - Theorems 3.4.1 and 3.4.2. In addition, we will prove that M and R are spaceable (Theorems 3.4.3 and 3.4.4), and we also recover the result proved in [3, Theorem 6.2] regarding the maximal density lineability of M and R. From the same theorems we will be able to infer that N and S are not spaceable.

3.2 The sets $B(v)$ and $A_0(v)$

In this chapter, all vector spaces are real, even though we are aware that our proofs would work on the complex number setting also in its present form or with a slight modification (some would only affect the real quantifier). When some non-trivial adaptation is required, as in Theorem 3.2.1 and Theorem 3.3.2, we will include a comment right after the proof. At last, but not less important, we should disclose that Theorem 3.2.1 may be of some independent interest. The reason why we may say this is because, to some experts in this field, the real challenge nowadays is to find interesting sets that are not lineable at all. Please see also Remark 3.2.3.

3.2.1 The non-lineability of $B(v)$

Theorem 3.2.1. *The set $B(v)$ is not lineable. More precisely, $B(v) \cup \{0\}$ only contains finite-dimensional subspaces of dimension* 1. *In particular, $A_d(v)$ is non-lineable for all* $0 < d < \infty$.

Proof. Let $z = (z_n)$, $y = (y_n) \in B(v)$, $d = \lim_{n\to\infty} |z_n/v_n|$ and $b = \lim_{n\to\infty} |y_n/v_n|$. We may assume without loss of generality that (z_n/v_n) contains a subsequence (z_{n_k}/v_{n_k}) which converges to d. If (y_{n_k}/v_{n_k}) converges to b, then

$$\lim_{k\to\infty} \left| \frac{bz_{n_k} - dy_{n_k}}{v_{n_k}} \right| = 0.$$

So, $(bz - dy) \notin B(v)$. On the other hand, if (y_{n_k}/v_{n_k}) - or at least a subsequence - converges to $(-b)$, then

$$\lim_{k\to\infty} \left| \frac{bz_{n_k} + dy_{n_k}}{v_{n_k}} \right| = 0.$$

Therefore, $B(v)$ is not lineable.

$\square$

Remark 3.2.1. The complex case follows from an application of the classical Bolzano-Weierstrass Theorem.

Remark 3.2.2. One may think that the assumption on the convergence of the sequence $(|w_n/v_n|)$ is somewhat a "too strong" condition. In order to clear the air, we should mention that with a slight modification of the arguments used in the last proof, one may prove that the set $\{w = (w_n) \in \ell_1 \mid 0 < \liminf |w_n/v_n| \leq \limsup |w_n/v_n| < \infty \text{ and } D(w) \text{ finite}\} \cup \{0\}$ only contains finite-dimensional subspaces of dimension 1.

Remark 3.2.3. There are very few known interesting examples of non-lineable sets. In [16] Gurariy proved that the set of continuous $\mathbb{R}$-valued functions on $[0,1]$ that attain its maximum at one (and only one) point is only 2-lineable. This result was generalized by González et. al in [17] in the following way: the set of continuous $\mathbb{R}$-valued functions on $\mathbb{R}$ that attains its maximum in exactly m points is also only 2-lineable for any $m \geq 1$. In [18] Aron and Hájek proved that for every infinite-dimensional real separable Banach space X and for every odd number n, $n \geq 3$, there is a polynomial $P : X \to \mathbb{R}$ which is n-homogeneous and for which $P^{-1}(0)$ is not lineable. Subsets that are not spaceable are also rare. An example of such set was given in Gurariy [1] - the set of everywhere differentiable functions on $[0,1]$.

3.2.2 Lineability of $A_0(v)$

Now, our goal is to show that $A_0(v)$ is $\mathfrak{c}$-lineable. To this end, we will need an auxiliary result.

Proposition 3.2.1. *The set $A_0(v)$ is non-empty. Furthermore, for a given w in $A_0(v)$, the set $L = \{w^k = (w_n^k) \mid k \in \mathbb{N}\}$ is a linearly independent subset of $A_0(v)$.*

Proof. For each $k \in \mathbb{N}$, let n_k be as in the proof of Proposition 3.1.1, and let us set $w = (w_n)$ where $w_n = 2^{-n_k}|v_n|$ if $n_k \leq n < n_{k+1}$ and $w_n = v_n$ otherwise. A direct inspection reveals that

$w \in A_0(v)$. Next, for each $k \in \mathbb{N}$, we set $w^k = (w_n^k)$. Let $L = \{w^k \mid k \in \mathbb{N}\} \subset A_0(v)$. We claim that L is a linearly independent set. Indeed, let $\lambda_1, \ldots, \lambda_r \in \mathbb{R}$ - not all zeros - satisfying

$$\sum_{j=1}^{r} \lambda_j w^{m_j} = 0.$$

Since, $w \in \ell_1$, we may find, for each $j = 1, 2, 3, \ldots, m_r$, entries $w_{n_{m_j}}$ of w pairwise distinct. If we set

$$M = \left[w_{n_{m_i}}^{j}\right]_{i,j=1}^{m_r} \text{ and } \vec{\beta} = [\beta_j]_{j=1}^{m_r},$$

then we may conclude that the system $M\vec{\beta} = \vec{0}$ admits a non-trivial solution. The latter contradicts the fact that the Vandermonde matrix M is invertible. So, the set L is linearly independent.

□

With an adaptation of the last proof, we may verify the following:

Theorem 3.2.2. *$A_0(v)$ is $\mathfrak{c}$-lineable. In particular, $B_0(v)$ is $\mathfrak{c}$-lineable.*

Proof. Let $w = (w_n) \in A_0(v)$, where $w_n > 0$ for all $n \in \mathbb{N}$. Let $D = \{w^\alpha = (w_n^\alpha) \mid \alpha \in (1,2)\} \subset A_0(v)$ - here we understand that $w_n^\alpha \in \mathbb{R}$. The proof boils down to showing the set D is linearly independent. Indeed, let $\lambda_1, \ldots, \lambda_r \in \mathbb{R}$ with

$$\sum_{j=1}^{r} \lambda_j w^{\gamma_j} = 0.$$

Let $\alpha_j \in \mathbb{Q} \cap (1,2)$ with $|\gamma_j - \alpha_j| < \delta$, where $\delta = (1/4)min_{1 \le i < j \le r}\{|\gamma_j - \gamma_i|\}$. Since $w \in \ell_1$, we may find, for each $j = 1, 2, 3, \ldots, r$, entries w_{n_j} of w pairwise distinct. By the choice of δ, we may infer that the α_j's are pairwise distinct. So, the matrix

$$\left[w_{n_i}^{\alpha_j}\right]_{i,j=1}^{r}$$

is invertible. For each $j \in \{1, \ldots, r\}$, let $\alpha_{j,k}$ be sequence contained in $\mathbb{Q} \cap (1,2)$ which converges to γ_j. Therefore, we see that the matrix

$$\left[w_{n_i}^{\gamma_j}\right]_{i,j=1}^{r}$$

is also invertible. Consequently $\lambda_j = 0$ for all $j = 1, \ldots, r$. □

3.2.3 Lineability of $A_\infty(v)$

We start this section with a technical lemma which will play a major role in the construction of a $\mathfrak{c}$-dense linear subspace contained in $A_\infty(v) \cup \{0\}$.

Lemma 3.2.1. *Let L be a subset of $A_\infty(v)$ fulfilling the following condition:*

(i) If $x = (x_n), y = (y_n) \in L$, then either $x \in A_\infty(y)$ or $y \in A_\infty(x)$.

Then L is a linearly independent subset of $A_\infty(v)$ and $\operatorname{span}(L) \subset A_\infty(v) \cup \{0\}$.

Proof. First of all, notice that if $x \in A_\infty(y)$ and $y \in A_\infty(z)$, then $x \in A_\infty(z)$. Indeed, we have

$$\lim_{n\to\infty}\left|\frac{x_n}{z_n}\right| = \lim_{n\to\infty}\left|\frac{x_n}{y_n}\right| \lim_{n\to\infty}\left|\frac{y_n}{z_n}\right| = \infty.$$

Next, let $x^{(1)} = (x_n^{(1)}), x^{(2)} = (x_n^{(2)}), \cdots, x^{(m)} = (x_n^{(m)})$ be arbitrary elements in L, and let $\alpha_1, \cdots, \alpha_m$ be non-zero real numbers. Based on the first paragraph, we may assume without loss of generality that $x^{(i)} \in A_\infty(x^{(j)})$ when $i < j$. Hence,

$$\lim_{n\to\infty}\left|\frac{\alpha_1 x_n^{(1)} + \alpha_2 x_n^{(2)} + \cdots + \alpha_m x_n^{(m)}}{v_n}\right| =$$
$$\lim_{n\to\infty}\left|\frac{\alpha_1 x_n^{(1)} + \alpha_2 x_n^{(2)} + \cdots + \alpha_m x_n^{(m)}}{x_n^{(1)}}\right|\left|\frac{x_n^{(1)}}{v_n}\right| = \infty,$$

since $\lim_{n\to\infty} x_n^{(j)}/x_n^{(1)} = 0$ for all $1 < j \le m$, and $\lim_{n\to\infty}|x_n^{(1)}/v_n| = \infty$. From the latter, we may conclude simultaneously that $\alpha_1 x^{(1)} + \alpha_2 x^{(2)} + \cdots + \alpha_m x^{(m)} \in A_\infty(v_n)$ and $\alpha_1 x_n^{(1)} + \alpha_2 x_n^{(2)} + \cdots + \alpha_m x_n^{(m)} \neq 0$ provided that $\alpha_1, \cdots, \alpha_m$ are non-zero scalars. Therefore, L is a linearly independent subset of $A_\infty(v)$. In particular, $\operatorname{span}(L) \subset A_\infty(v) \cup \{0\}$. □

Theorem 3.2.3. *$A_\infty(v)$ is $\mathfrak{c}$-dense-lineable. In particular, $X(v)$ is $\mathfrak{c}$-dense-lineable.*

Proof. Let α be in $(0, 2)$. For each $k \in \mathbb{N}$, let n_k be as in the proof of Proposition 3.1.1. After repeating the same arguments presented in the proof of Proposition 3.1.1, we may conclude that the sequence $x^{(\alpha)} = (x_n^{(\alpha)}) \in A_\infty(v)$, where $x_n^{(\alpha)} = 2^{\alpha k} v_n$ if $n_k \le n < n_{k+1}$ and $x_n^{(\alpha)} = v_n$ otherwise. In additon, if $\alpha, \beta \in (0, 2)$ and $\alpha > \beta$, then $\lim_{n\to\infty}|x_n^{(\alpha)}/x_n^{(\beta)}| = \lim_{k\to\infty} 2^{(\alpha-\beta)k} = \infty$. Hence, if we set $L = \{x^{(\alpha)} \mid \alpha \in (0, 2)\}$, we see that L satisfies the condition (i) in Lemma 3.2.1. Therefore, $\operatorname{span}(L)$ is a subset of $A_\infty(v) \cup \{0\}$ and $\dim \operatorname{span}(L) = \operatorname{card}(L) = \mathfrak{c}$.

At this point, as the reader may have noticed, we already have established the $\mathfrak{c}$-lineability of $A_\infty(v)$. Now, let us see how we can modify the set L in order to obtain a dense subspace of ℓ_1 contained in $A_\infty(v)$. Indeed, fix $r = (r_1, r_2, \cdots, r_m, 0, 0, \cdots) \in c_{00}(\mathbb{Q})$ and choose $M_r = \{y^{(k)} \mid k \in \mathbb{N}\}$ to be a countable subset of L. For each $k \in \mathbb{N}$, we take $n_k \in \mathbb{N}$ such that $n_k > m$ and $\sum_{n=n_k}^{\infty}\left|y_n^{(k)}\right| < 1/k$. Now, we set $z^{(k)} = (z_n^{(k)})$ where $z_n^{(k)} = r_n$ if $1 \le n \le m$, $z_n^{(k)} = 0$ if $m < n < n_k$ and $z_n^{(k)} = y_n^{(k)}$ if $n \ge n_k$. By construction, $z^{(k)} \in A_\infty(x)$ (resp. $x \in A_\infty(z^{(k)})$) if and only if $y^{(k)} \in A_\infty(x)$ (resp. $x \in A_\infty(y^{(k)})$). Consequently, the set $N_r = \{z^{(k)} \mid k \in \mathbb{N}\}$ also satisfies the condition (i) in Lemma 3.2.1. On the other hand, from the inequality

$$\|z^{(k)} - r\| = \sum_{n=n_k}^{\infty}\left|y_n^{(k)}\right| < \frac{1}{k},$$

one may infer that $\lim_{k\to\infty} z^{(k)} = r$. So, $r \in \overline{N_r}$.

Next, for each $s \in c_{00}(\mathbb{Q})$ we choose M_s to be a countable subset of L with $M_t \cap M_s = \varnothing$ whenever $t, s \in c_{00}(\mathbb{Q})$ and $t \neq s$. Repeating the previous construction, for each $s \in c_{00}(\mathbb{Q})$, we obtain a set N_s where $s \in \overline{N_s}$ and which also satisfies the condition (i) in Lemma 3.2.1. Set

$$L' = \left(L \setminus \left(\bigcup_{r \in c_{00}(\mathbb{Q})} M_r \right) \right) \bigcup_{r \in c_{00}(\mathbb{Q})} N_r.$$

Then, $\mathrm{card}(L') = \mathfrak{c}$. By Lemma 3.2.1, L' is a linearly independent set and $\mathrm{span}(L') \subset A_\infty(v) \cup \{0\}$. Since, $r \in \overline{N_r} \subset \overline{\mathrm{span}(L')}$ for each $r \in c_{00}(\mathbb{Q})$, we see that $\ell_1 = \overline{\mathrm{span}(L')}$. Thus, $A_\infty(v)$ is $\mathfrak{c}$-dense-lineable. $\square$

After revisiting Theorem 3.2.2 under the light of the methods employed in the proof of the last theorem, we see that it now admits the following form:

Theorem 3.2.4. *$A_0(v)$ is $\mathfrak{c}$-dense-lineable. In particular, $B_0(v)$ is $\mathfrak{c}$-dense-lineable.*

Remark 3.2.4. It is folklore that $\dim \ell_1 = \mathfrak{c}$. For this reason, one may replace the quantifier "$\mathfrak{c}$-dense-lineable" with "maximal dense-lineable" in all the results presented in the last two sections.

3.2.4 Topological classification

Let (X, τ) be a topological space. A subset Y of X is called a meager subset of X if Y can be written as a countable union of nowhere dense subsets of X - a set is nowhere dense if its closure has an empty interior. The main purpose of this section is to prove that the sets $A_\infty(v)$, $B(v)$ and $B_0(v)$ are all meager sets. We start with a couple of lemmas.

Lemma 3.2.2. *For each $r > 0$ and $k \in \mathbb{N}$, the set $W_r^k(v) = \{w = (w_n) \in \ell_1 \mid |w_n| \geq r|v_n| \text{ for all } n > k\}$ is closed and has empty interior. In particular, $\bigcup_{\substack{r \in \mathbb{Q}^+ \\ k \in \mathbb{N}}} W_r^k(v)$ is a meager set.*

Proof. Let $(z^{(l)})_{l \in \mathbb{N}} \subset W_r^k(v)$ be such that $\lim_{l \to \infty} z^{(l)} = (z_n)$. Then, for each $n > k$ and for all $l \in \mathbb{N}$, we have the inequality $|z_n^{(l)}| \geq r|v_n|$. Consequently, $|z_n| \geq r|v_n|$ for all $n > k$. Hence, $z \in W_r^k(v)$. For the second part, let $w = (w_n) \in W_r^k(v)$ and $\epsilon > 0$. We choose $n_0 \geq k$ satisfying $|w_n| < \epsilon$ and $|v_n| > 0$ for all $n \geq n_0$. Next, we set $\zeta = (\zeta_n)$ with $\zeta_n = w_n$ if $n \neq n_0$ and zero otherwise. On the one hand, $|\zeta_{n_0}| = 0 < r|v_{n_0}|$. So, $\zeta \notin W_r^k(v)$. On the other hand, $\|\zeta - w\| = |w_{n_0}| < \epsilon$. Therefore, w does not lie in the interior of $W_r^k(v)$. $\square$

Lemma 3.2.3. *For each $r > 0$ and $k \in \mathbb{N}$, the set $U_r^k(v) = \{w = (w_n) \in \ell_1 \mid |w_n| \leq r|v_n| \text{ for all } n > k\}$ is closed and has empty interior. In particular, $\bigcup_{\substack{r \in \mathbb{Q}^+ \\ k \in \mathbb{N}}} U_r^k(v)$ is a meager set.*

Proof. The first part follows the same steps as in the previous lemma. Now, let $w \in U_r^k(v)$ and $\epsilon > 0$. Let $n_0 \in \mathbb{N}$ be such that $n_0 > k$ and $r|v_{n_0}| < \epsilon/2$. Set $\zeta = (\zeta_n)$ where $\zeta_n = w_n$ if

$n \neq n_0$ and $\epsilon/2$ otherwise. So, $|\zeta_{n_0}| > r|v_{n_0}|$ and

$$\|\zeta - w\| = |\zeta_{n_0} - w_{n_0}| \leq |\zeta_{n_0}| + |w_{n_0}| \leq \epsilon/2 + r|v_{n_0}| < \epsilon.$$

Therefore, $\zeta \notin U_r^k(v)$ and $\|\zeta - w\| < \epsilon$. The latter allows us to conclude that the interior of $U_r^k(v)$ is empty. □

Theorem 3.2.5. *$A_\infty(v)$ and $B(v)$ are meager sets.*

Proof. Let $w \in A_\infty(v) \cup B(v)$. So, there exist $r \in \mathbb{Q}^*$ and $k \in \mathbb{N}$ such that $|w_n/v_n| \geq r$ for all $n > k$. Hence, $w \in W_r^k(v)$. Then, $A_\infty(v), B(v) \subset \bigcup_{\substack{r \in \mathbb{Q}^+ \\ k \in \mathbb{N}}} W_r^k(v)$. □

Theorem 3.2.6. *$B_0(v)$ is a meager set and $X(v)$ is residual. In particular, $(\ell_1 \setminus X(v))$ is a meager set.*

Proof. Let $w \in (\ell_1 \setminus X(v)) \cup A_d(v)$ for $d < \infty$. Then, there exist $r \in \mathbb{Q}^*$ and $k \in \mathbb{N}$ such that $|w_n/v_n| < r$ for all $n > k$. Thus, $w \in U_r^k(v)$. Hence, $(\ell_1 \setminus X(v)), B_0(v) \subset \bigcup_{r \in \mathbb{Q}^+, k \in \mathbb{N}} U_r^k(v)$ is a meager set. Finally, since $(\ell_1 \setminus X(v))$ is meager, it follows that $X(v)$ is a residual set. □

3.3 The set $X(v)$

Theorem 3.3.1 below, which states that every infinite-dimensional closed subspace of ℓ_1 has a nonzero element with infinitely many zeros, was originally proved in [5, Corollary 3.4]. Nevertheless, we have decided to present a new proof - based on the original one - which in our humble opinion is more direct. This simplification will allow us to generalize this outstanding result in the following way: Every infinite-dimensional closed subspace of ℓ_1 contains an element of $X(v)$ (Theorem 3.3.2). At the end we will also show that $X(v)$ is maximal spaceable (Theorem 3.3.3).

Theorem 3.3.1. *Every infinite-dimensional closed subspace of ℓ_1 has a non-zero element with infinitely many zeros.*

Proof. Let W be an infinite-dimensional closed subspace of ℓ_1 and let $\zeta = (\zeta_n)$ be an element of W. Call $D(\zeta) = \{n \in \mathbb{N} \mid \zeta_n = 0\}$ and $E(\zeta) = \{n \in \mathbb{N} \mid \zeta_n \neq 0\}$. If $D(\zeta)$ is infinite for some non-zero vector $\zeta \in W$ then the proof is over. Let us fix $z^{(1)} = (z_n^{(1)}) \in W$, and we may assume without loss of generality that $D(z^{(1)})$ is finite. For each natural number m, one may see that the subspace $Y_m = \{w = (w_n) \in W \mid w_n = 0 \text{ if } n \leq m\}$ is also infinite-dimensional. Now, let us choose $m_1 \in \mathbb{N}$ such that $\sum_{n=m_1+1}^{\infty} |z_n^{(1)}| < 1/2$ and $|z_{m_1}^{(1)}| \neq 0$. Let $y^{(1)}$ be an element of Y_{m_1} with $\|y^{(1)}\| = 1$. Next, let us take $k_1 > m_1$ such that $|z_{k_1}^{(1)}| < \dfrac{|y_{k_1}^{(1)}|}{2}$, and we set $z^{(2)} = z^{(1)} + x^{(1)} = z^{(1)} + \left(-y^{(1)} \dfrac{z_{k_1}^{(1)}}{y_{k_1}^{(1)}} \right)$. In particular, $z_{k_1}^{(2)} = 0$ and $\|z^{(2)} - z^{(1)}\| = \|x^{(1)}\| < 1/2$, and $z_{m_1}^{(2)} = z_{m_1}^{(1)} \neq 0$.

For the next step, we may suppose that $D(z^{(2)})$ is finite, otherwise, we are done. Now, we choose m_2 such that $0 < \sum_{n=m_2+1}^{\infty} |z_n^{(2)}| < 1/2^2$ and $m_2 > k_1 > m_1$. As in the previous paragraph, we take $y^{(2)}$ in Y_{m_2} with $\|y^{(2)}\| = 1$ and we set $z^{(3)} = z^{(2)} + x^{(2)} = z^{(2)} + \left(-\frac{z_{k_2}^{(2)}}{y_{k_2}^{(2)}} y^{(2)}\right)$, where $k_2 > m_2$ and $0 < |z_{k_2}^{(2)}| < \frac{|y_{k_2}^{(2)}|}{2^2}$. Hence, $z_{k_1}^{(3)} = z_{k_2}^{(3)} = 0$ and $\|z^{(3)} - z^{(2)}\| = \|x^{(2)}\| < 1/2^2$, and $z_{m_1}^{(3)} = z_{m_1}^{(2)} = z_{m_1}^{(1)} \neq 0$. So, inductively, we construct the sequences $(y^{(l)})_l$ and $(z^{(l)})_l$ such that

$$\|z^{(l+1)} - z^{(l)}\| = \|x^{(l)}\| = \left\| -\frac{z_{k_l}^{(l)}}{y_{k_l}^{(l)}} y^{(l)} \right\| < 1/2^l,$$

where $y^{(l)} \in Y_{m_l}$ (with $\|y^{(l)}\| = 1$) and $z_{k_j}^{(l)} = 0$ for each $j = 1, 2, \ldots, (l-1)$ provided that $D(z^{(p)})$ is a finite set for each $l \in \mathbb{N}$. In addition, $z_{m_1}^{(l)} = z_{m_1}^{(1)} \neq 0$ for all $l \in \mathbb{N}$. It is conspicuous to see that $(z^{(l)})_l$ is a Cauchy sequence contained in W, therefore converges to an element $\delta = (\delta_n) \in W$, since W is closed. By construction, $\delta_{m_1} = z_{m_1}^{(1)} \neq 0$ and $\delta_{k_l} = 0$ for all $l \in \mathbb{N}$. The proof is now complete. □

Theorem 3.3.2. *Every infinite-dimensional closed subspace of ℓ_1 contains an element $z \in X(v)$.*

Proof. Let W be an infinite-dimensional closed subspace of ℓ_1. For each $m \in \mathbb{N}$, let Y_m be as in the proof of Theorem 3.3.1. Choose $m_1 \in \mathbb{N}$ and $y^{(1)} \in Y_{m_1}$ where $\sum_{n=m_1+1}^{\infty} |v_n| < 1/2^2$ and $\|y^{(1)}\| = 1$. Next, let us take $k_1 > m_1$ with $|y_{k_1}^{(1)}| > 2^2 |v_{k_1}|$, and set $x^{(1)} = (1/2) \cdot y^{(1)}$ and $z^{(1)} = x^{(1)}$. Clearly, $\|z^{(1)}\| = 1/2$ and $|z_{k_1}^{(1)}| > 2|v_{k_1}|$.

The element $z^{(2)}$ will be constructed in the following way. Pick $m_2 > k_1$ and $y^{(2)} \in Y_{m_2}$ where $\sum_{n=m_2+1}^{\infty} |v_n| < 1/2^4$ and $\|y^{(2)}\| = 1$. As before, we choose $k_2 > m_2$ such that $|y_{k_2}^{(2)}| > 2^4 |v_{k_2}|$, and we set $x^{(2)} = (1/2^2) \cdot y^{(2)}$ and $z^{(2)} = x^{(2)} + z^{(1)}$. Here, we may assume without loss of generality that the entries $x_{k_2}^{(2)}$ and $z_{k_2}^{(1)}$ have the same sign - if this is not the case, replace $x^{(2)}$ with $-x^{(2)}$. That being said, we have $\|z^{(2)} - z^{(1)}\| = \|x^{(2)}\| = 1/2^2$, and $|z_{k_2}^{(2)}| \geq |x_{k_2}^{(2)}| > 2^2 |v_{k_2}|$. In addition, since $m_2 > k_1$, we also have $|z_{k_1}^{(2)}| = |z_{k_1}^{(1)}| \geq 2|v_{k_1}|$.

Proceeding by induction, we may find a sequence $(z^{(l)})_l$ in W and an increasing sequence (k_s) in $\mathbb{N}$ fulfilling the following two conditions:

(i) $\|z^{(l)} - z^{(l-1)}\| = \|x^{(l)}\| = 1/2^l$. In particular, $(z^l)_l$ is a Cauchy sequence. Therefore, there exists $z \in W$ such that $z = \lim_{l \to \infty} z^{(l)}$.

(ii) For each fixed $s \in \mathbb{N}$, we have $|z_{k_s}^{(l)}| > 2^s |v_{k_s}|$ for all $l \geq s$. Hence, $|z_{k_s}| \geq 2^s |v_{k_s}|$ for each $s \in \mathbb{N}$. Thus, $z \in X(v)$.

Then, we conclude that $z \in (W \cap X(v))$. The proof is now complete. □

Remark 3.3.1. The last proof contains an argument where we use the fact that we are considering real numbers (when we assume that $x_{k_2}^{(2)}$ and $z_{k_2}^{(1)}$ have the same sign). For the complex case, we can adapt this step by assuming $x_{k_2}^{(2)}$ and $z_{k_2}^{(1)}$ have the same argument, otherwise, we can multiply $x^{(2)}$ by $e^{i\theta}$ for a suitable θ, preserving the norm of $x^{(2)}$. Then we have $|z_{k_1}^{(2)}| = |z_{k_1}^{(1)}| \geq 2|v_{k_1}|$, as we wanted. The same process can be made inductively to construct $(z^{(l)})_l$.

We close this section with the following result.

Theorem 3.3.3. $X(v)$ *is maximal spaceable.*

Proof. Let $(N_k)_{k\in\mathbb{N}}$ be a partition of $\mathbb{N}$ where each N_k is an infinite set. Now, we fix an element $w \in A_\infty(v)$. Next, we set $w^{(k)} = (w_n^{(k)})$ where $w_n^{(k)} = w_n$ if $n \in N_k$ and $w_n^{(k)} = 0$ otherwise. Clearly $w^{(k)} \in X(v)$ for each $k \in \mathbb{N}$. We claim that $W = \operatorname{span}(\{w^{(k)} \mid k \in \mathbb{N}\})$ is a subspace in $X(v) \cup \{0\}$ and $\overline{W} \subset X(v) \cup \{0\}$. Indeed, let $\alpha_1, \cdots, \alpha_m \in \mathbb{R}$ and $k_1, \cdots, k_m \in \mathbb{N}$. We may assume that $\alpha_1 \neq 0$. So,

$$\sup_{n\in D(v)} \frac{\left|\alpha_1 w_n^{(k_1)} + \alpha_2 w_n^{(k_2)} + \cdots + \alpha_m w_n^{(k_m)}\right|}{|v_n|} \geq$$

$$\sup_{n\in(N_{k_1}\cap D(v))} \frac{\left|\alpha_1 w_n^{(k_1)} + \alpha_2 w_n^{(k_2)} + \cdots + \alpha_m w_n^{(k_m)}\right|}{|v_n|} = \sup_{n\in(N_{k_1}\cap D(v))} \frac{|\alpha_1 w_n|}{|v_n|} = \infty.$$

Thus, $\alpha_1 w_{k_1} + \cdots + \alpha_m w_{k_m} \in X(v)$ and $\{w^{(k)} \mid k \in \mathbb{N}\}$ is a linearly independent set. Thus, W is an infinite-dimensional subspace in $X(v) \cup \{0\}$. It remains to show that $\overline{W} \subset X(v) \cup \{0\}$. But first, let us observe that the condition x in W implies that $x_n = \alpha_k w_n$ for each $n \in N_k$, where $\alpha_k \in \mathbb{R}$ depends on k.

Finally, let us suppose that $x \neq 0$ is the limit of a sequence $(y^l)_{l\in\mathbb{N}} \subset W$. Let $m \in \mathbb{N}$ be such that $x_m \neq 0$, and $k \in \mathbb{N}$ be such that $m \in N_k$. Note that $y_m^{(l)} = \alpha_k^{(l)} w_m$, and $\lim_{l\to\infty} \alpha_k^{(l)} w_m = x_m$. In particular, $w_m \neq 0$. So, $\alpha = \lim_{l\to\infty} \alpha_k^{(l)} = x_m/w_m \neq 0$. Let us recall that for each $n \in N_k$, we have $y_n^{(l)} = \alpha_k^{(l)} w_n$. Hence, $x_n = \lim_{l\to\infty} \alpha_k^{(l)} w_n = \alpha w_n$ for each $n \in N_k$. Thus,

$$\sup_{n\in D(v)} \frac{|x_n|}{|v_n|} \geq \sup_{n\in N_k} \frac{|\alpha||w_n|}{|v_n|} = \infty.$$

Therefore, $x \in X(v)$. Then, $\overline{W} \subset X(v) \cup \{0\}$. As the minimum dimension for an infinite-dimensional Banach space is $\mathfrak{c}$, then $\overline{W}$ has maximal dimension, which completes our proof. □

3.4 Applications

3.4.1 Immediate applications

As applications of Theorem 3.3.1 and Theorem 3.3.2 we may derive, respectively, the next two results.

Theorem 3.4.1. *The set $A_\infty(v)$ is not spaceable.*

Theorem 3.4.2. *The set $B_0(v)$ is not spaceable.*

Remark 3.4.1. Theorem 3.3.2 guarantees the non-spaceability of $B_0(v)$ even if we decide to remove the finiteness condition on the kernel $D(w)$ - for $w \in A_0(v)$ - that appeared in the definition of the set $A_0(v)$.

3.4.2 Applications to other subsets of ℓ_1

We recommend the reader to see the introduction to remember the meaning of the sets M, N, R and S.

Theorem 3.4.3. *The following assertions hold:*

(*i*) *M is maximal dense-lineable.*

(*ii*) *N is maximal dense-lineable.*

(*iii*) *M is spaceable.*

(*iv*) *N is not spaceable.*

(*v*) *M is residual.*

(*vi*) *N is meager.*

Proof. Let $v = (v_n) \in M$ where $\lim_{n\to\infty} |v_n|^{1/n} = 1$ and $v_n > 0$ for all $n \in \mathbb{N}$. For this given element, we consider the set $X(v)$. Now, we fix an element $w = (w_n) \in X(v)$. In the sequence, we will show that $w \in M$. Indeed, since $w \in \ell_1$, then $\limsup_{n\to\infty} |w_n|^{1/n} \leq 1$. For a given $\epsilon > 0$ there exists $m_0 \in \mathbb{N}$ such that $|v_n|^{1/n} > (1-\epsilon)$ for all $n \geq m_0$. On the other hand, since $w \in X(v)$, there exists $n_0 > m_0$ such that $|w_{n_0}| > |v_{n_0}|$. So, $|w_{n_0}|^{1/n_0} > |v_{n_0}|^{1/n_0} > 1-\epsilon$. Therefore, $\limsup_{n\to\infty} |w_n|^{1/n} = 1$. Thus, $X(v) \subset M$. Since $X(v)$ is maximal dense-lineable (Theorem 3.2.3), maximal spaceable (Theorem 3.3.3) and residual (Theorem 3.2.6), we have (i), (iii) and (v). Next, by Theorem 3.3.2, for a given infinite-dimensional closed subspace W of ℓ_1, we have $W \cap X(v) \neq \varnothing$, and because $X(v) \subset M$, we see that $W \not\subset N$. Therefore, N is not spaceable, which proves (iv).

Now, we fix $v = (v_n) \in N$, where $v_n > 0$ for all $n \in \mathbb{N}$. Let us consider the set $A_0(v)$. We will prove (ii). If $w = (w_n) \in A_0(v)$, then $|w_n| < |v_n|$ for all $n \geq n_0$. The latter implies that $\limsup_{n\to\infty} |w_n|^{1/n} \leq \limsup_{n\to\infty} |v_n|^{1/n} < 1$, that is, $w \in N$. Then, $A_0(v) \subset N$. From the fact that $A_0(v)$ is maximal dense-lineable (Theorem 3.2.4), we may infer that N is also maximal dense-lineable, and this completes the proof of (ii).

Part (vi) follows from the observation that $N = (\ell_1 \backslash M)$, and M is residual (by (v)). ☐

By similar arguments, we may prove:

Theorem 3.4.4. *The following assertions hold:*

(i) *R is maximal dense-lineable.*

(ii) *S is maximal dense-lineable.*

(iii) *R is spaceable.*

(iv) *S is not spaceable;*

(v) *R is residual.*

(vi) *S is meager.*

Proof. As in [3], we adopt the conventions $0/0 = 0$ and $a/0 = \infty$ for every $a \in (0, \infty)$. We recall that, for a strictly positive sequence (a_n), it is always true the following estimate

$$\liminf_{n\to\infty} \frac{a_{n+1}}{a_n} \leq \liminf_{n\to\infty} a_n^{1/n} \leq \limsup_{n\to\infty} a_n^{1/n} \leq \limsup_{n\to\infty} \frac{a_{n+1}}{a_n}. \tag{3.1}$$

Now, let us fix $(v_n) \in \ell_1$ with $\lim_{n\to\infty} |v_n|^{1/n} = 1$ and $v_n > 0$ for all n. As in the last theorem, we have $X(v) \subset M$. Given $w = (w_n) \in X(v)$ we have two possibilities: $D(w)$ (the kernel of w) is infinite or $D(w)$ is finite. In the case that $D(w) = \{n \in \mathbb{N} \mid w_n \neq 0\}$ is infinite, we see that the set $\{n \in \mathbb{N} \mid w_{n+1} \neq 0 \text{ and } w_n = 0\}$ is also infinite. Thus, by our convention, $\limsup |w_{n+1}/w_n| = \infty$, that is, $w \in R$. In the second case, we can apply (3.1) in order to conclude that $\limsup_{n\to\infty} |w_{n+1}/w_n| \geq \limsup_{n\to\infty} |w_n|^{1/n} = 1$. So, $w \in R$. Hence, $X(v) \subset R$. Consequently, R is maximal dense-lineable, spaceable, and residual. By Theorem 3.3.2, every infinite-dimensional closed subspace W of ℓ_1 intercepts $X(v)$. Since $X(v) \subset R$, S is not spaceable. A direct consequence of R being residual is that S is meager.

At last let us prove (ii). For the element $w = (w_n) = (1/2^n) \in S$, we have that $(w_n^\alpha) \in S$ for all $\alpha \in (1, 2)$. Given $\alpha_1 < \cdots < \alpha_k \in (1, 2)$ and $\lambda_1, \cdots, \lambda_k$ non-zero scalars, we have

$$\lim_{n\to\infty} \frac{\lambda_1 w_{n+1}^{\alpha_1} + \cdots + \lambda_k w_{n+1}^{\alpha_k}}{\lambda_1 w_n^{\alpha_1} + \cdots + \lambda_k w_n^{\alpha_k}} = \lim_{n\to\infty} \frac{\lambda_1 2^{-(n+1)\alpha_1} + \cdots + \lambda_k 2^{-(n+1)\alpha_k}}{\lambda_1 2^{-n\alpha_1} + \cdots + \lambda_k 2^{-n\alpha_k}} =$$

$$\lim_{n\to\infty} \frac{\lambda_1 2^{-\alpha_1} + \cdots + \lambda_k 2^{n(\alpha_1-\alpha_k)-\alpha_k}}{\lambda_1 + \cdots + \lambda_k 2^{n(\alpha_1-\alpha_k)}} = 2^{-\alpha_1} < 1.$$

Therefore, $\mathrm{span}(\{(w_n^\alpha) \mid \alpha \in (1, 2)\})$ is contained in S. It is also easy to see that the elements of this set are non-trivial in the sense that they have infinite support. Now, we use the same technique employed in the proof of Theorem 3.2.3 to conclude that S is maximal dense-lineable. $\square$

4 Lineability in measurable function spaces

Let $(\mathscr{M}, \mathscr{A}, \mu)$ be a measure space, where μ is a positive measure. We write $L_0(\mathscr{M}) = L_0(\mathscr{M}, \mathscr{A}, \mu)$ to represent the set of all measurable functions $f : \mathscr{M} \to \mathbb{R}$. It is well-known that $L_p(\mathscr{M}) = L_p(\mathscr{M}, \mathscr{A}, \mu) := \{f \in L_0(\mathscr{M}) \mid \int_{\mathscr{M}} |f|^p d\mu < \infty\}$ $(1 \leq p < \infty)$ and $L_\infty(\mathscr{M}) = L_\infty(\mathscr{M}, \mathscr{A}, \mu) := \{f \in L_0(\mathscr{M}) \mid \ |f(t)| \leq C \ \text{ a.e. where } \ C = C(f) \geq 0\}$ are Banach spaces when equipped respectively with the norms $\|f\|_p = \int_{\mathscr{M}} |f|^p d\mu$ and $\|g\|_\infty = \inf\{C \geq 0 \mid |g(t)| \leq C \ \ a.e.\}$ $(f \in L_p(\mathscr{M})$ and $g \in L_\infty(\mathscr{M})$.) In addition, for each set $A \in \mathscr{A}$, the symbol χ_A denotes the characteristic function on A, that is, $\chi_A(t) = 1$ if $t \in A$ and zero otherwise - note that $\chi_A \in L_0(\mathscr{M})$.

In the previous chapter, for the measure space $(\mathbb{N}, \mathscr{P}(\mathbb{N}), \mu)$ with μ being the counting measure and $\mathscr{P}(\mathbb{N})$ as the power set of $\mathbb{N}$, we have considered questions regarding the lineability (spaceability) of certain subsets of $\ell_1 = L_1(\mathbb{N}, \mathscr{P}(\mathbb{N}), \mu)$. Now, we will extend some of these results for a more general measure space $(\mathscr{M}, \mathscr{A}, \mu)$. However, in order to make everything work out we need to overcome some technicalities such as (in the ℓ_1 setting the three questions below can be easily answered - note that for the third one, we have $\dim_{\mathbb{R}} \ell_1 = \mathfrak{c}$):

- How can we consider the notion of limit at infinity?
- How can we guarantee the existence of a strictly positive function $f \in L_1(\mathscr{M})$?
- When is $L_p(M)$ infinite-dimensional? What is $\dim_{\mathbb{R}}(L_p(\mathscr{M}))$?

In Section 4.1, we introduce the notion of convergence in measure at infinity - for a very large class of sets $\mathscr{M}$ - and we also provide necessary and sufficient conditions in order to guarantee the uniqueness of such limits (which we can not take for granted). In Section 4.2 we provide some characterizations of when $L_p(\mathscr{M})$ is infinite-dimensional (Theorem 4.2.1 and Corollary 4.2.1) in terms of $\mathscr{M}$ and μ. Besides that, we provide a formula to compute $\dim_{\mathbb{R}}(L_p(\mathscr{M}))$ whenever $L_p(\mathscr{M})$ is infinite-dimensional (Theorem 4.2.2). In Section 4.3, we completely characterize measure spaces $(\mathscr{M}, \mathscr{A}, \mu)$ that admit p-integrable strictly positive functions (Theorem 4.3.1). We also show in Theorem 4.3.2 that the problems that were considered in the last chapter for the space ℓ_1 can also be well-posed for $L_1(\mathscr{M}, \mathscr{A}, \mu)$ whenever $(\mathscr{M}, \mathscr{A}, \mu)$ belongs to a certain class of measure spaces (which happens to be a very large class). As in the last chapter, we deal with real vector spaces, but the results can be adapted to the complex case.

4.1 Convergence in measure at infinity and the main problem

In this section, we understand that $(\mathscr{M}, \mathscr{A}, \mu)$ is a measure space, where $\mathscr{M} = (\mathscr{M}, +, \|\cdot\|)$ is an unbounded subset of a fixed real normed vector space $Y = (Y, +, \|\cdot\|)$,

$\mathscr{A}$ is the Borel σ-algebra in $\mathscr{M}$ ($\mathscr{A}$ is the σ-algebra generated by the open sets of $\mathscr{M}$ when $\mathscr{M}$ is endowed with the subspace topology induced by the normed space Y) and μ is an arbitrary positive measure. In addition, from now on, we will adopt the convention: $B_\zeta(t) = \{z \in \mathscr{M} \mid \|z - t\| < \zeta\}$, and $B_\zeta(t)^c = \{z \in \mathscr{M} \mid \|z - t\| \geq \zeta\}$.

4.1.1 Convergence in measure at infinity and ULP_∞-measures

We start with a preliminary result that will be the starting point of our investigation.

Proposition 4.1.1. *Let $f \in L_1(\mathscr{M})$. Then, for all $\epsilon, \delta > 0$, there exists $\zeta > 0$ where the set $A_{\epsilon,f}(\zeta) = A_\epsilon(\zeta) = \{t \in B_\zeta(0)^c \mid |f(t)| > \epsilon\}$ has measure less than δ.*

Proof. Fix $\epsilon > 0$. Set $A_\epsilon = \{y \in \mathscr{M} \mid |f(y)| > \epsilon\}$, and let $A_\epsilon(\zeta)$ be as in the statement. Clearly, $A_\epsilon(\zeta) = A_\epsilon \cap B_\zeta(0)^c$. For each $n \in \mathbb{N}$, we consider $f_n = \chi_{B_n(0)}|f|$. Now, fix $\delta > 0$, and let us suppose that $\mu(A_\epsilon(\zeta)) \geq \delta$ for all $\zeta > 0$. Hence,

$$\begin{aligned}\int_{\mathscr{M}} |f| d\mu - \int_{\mathscr{M}} f_n d\mu &= \int_{\mathscr{M}} |f| \chi_{B_n(0)^c} d\mu = \int_{B_n(0)^c} |f| d\mu \\ &= \int_{B_n(0)^c \cap A_\epsilon} |f| d\mu + \int_{B_n(0)^c \setminus A_\epsilon} |f| d\mu \\ &\geq \epsilon \cdot \mu(A_\epsilon(n)) \geq \epsilon\delta.\end{aligned}$$

The latter inequality contradicts the Monotone Convergence Theorem. The proof is now complete. □

The last result motivates us to introduce the following definition:

Definition 4.1.1. Let $f \in L_0(\mathscr{M})$. We say that f converges in measure to a real number d at infinity if for all $\epsilon, \delta > 0$ there exists $\zeta > 0$ such that the set $C_{\epsilon,f}(\zeta) = C_\epsilon(\zeta) = \{s \in B_\zeta(0)^c \mid |f(s) - d| > \epsilon\}$ has measure less than δ. In this case we will write

$$\mu\text{-}\lim_{t\to\infty} f(t) = d.$$

In the case that $d = \infty$, we require that for all $N, \delta > 0$ there exists $\zeta > 0$ where the set $D_{N,f}(\zeta) = D_N(\zeta) = \{s \in B_\zeta(0)^c \mid f(s) < N\}$ has measure less than δ. The case $d = -\infty$ is defined in the same fashion.

At this stage a question presents itself: Is the limit in measure at infinity unique? The answer to this question is negative in general as we can see in the next example.

Example 4.1.1. In this paragraph, $\mathscr{M} = Y = \mathbb{R}$ is considered with the usual topology induced by the absolute value, and μ is the Lebesgue measure on the Borel σ-algebra $\mathscr{A}$. The function $f : \mathbb{R} \to \mathbb{R}$ defined as $f(t) = e^{-t^2}$ for all $t \in \mathscr{M}$ is integrable with respect to the measure μ. Hence, the map $\nu : \mathscr{A} \to \mathbb{R} \cup \{\infty\}$ with $\nu(E) = \int_E f\, d\mu$ for each $E \in \mathscr{A}$ defines a finite measure on the σ-algebra $\mathscr{A}$. In particular, $\lim_{t\to\infty} \nu(B_t(0)^c) = 0$. Moreover, any constant function lies in $L_1(\mathscr{M}, \mathscr{A}, \nu)$. Now, let g be a constant function. We claim that $\mu\text{-}\lim_{t\to\infty} g(t) = d$ for all $d \in \mathbb{R}$. Indeed, let $\epsilon, \delta > 0$. We choose ζ such that $\nu(B_\zeta(0)^c) < \delta$. Therefore, the set $C_{\epsilon,g}(\zeta) = \{s \in B_\zeta(0)^c \mid \ |g(s) - d| > \epsilon\}$ has measure less than δ, since $C_{\epsilon,g}(\zeta) \subset B_\zeta(0)^c$. This concludes the proof of our claim.

The last example gives us a taste of how pathological is the behavior of the convergence in measure at infinity. That being said, we see ourselves compelled to introduce the following definition.

Definition 4.1.2. We say that the measure μ possesses the Uniqueness Limit Property at infinity (for short μ is a ULP_∞-measure) when every measurable function f admits at most one limit in measure at infinity.

The next result characterizes ULP_∞-measures.

Theorem 4.1.1. *μ is a ULP_∞-measure if and only if $\mu(B_\zeta(0)^c) = \infty$ for all $\zeta \geq 0$.*

Proof. Let us suppose that μ is a ULP_∞-measure with $\mu(B_\zeta(0)^c) = L < \infty$ for some $\zeta > 0$. Consequently the function $\chi_{B_{\zeta,\gamma}(t)} \in L_1(\mathscr{M})$, where $B_{\zeta,\gamma}(t) = \{z \in \mathscr{M} \mid \zeta \leq \|z - t\| < \gamma\}$, for all $\gamma > \zeta$. By the Monotone Convergence Theorem,

$$\mu(B_\zeta(0)^c) = \int_{B_\zeta(0)^c} 1 d\mu = \int_{\mathscr{M}} \chi_{B_\zeta(0)^c} d\mu = \lim_{n\to\infty} \int_{\mathscr{M}} \chi_{B_{\zeta,\zeta+n}(0)} d\mu.$$

The latter yields $\lim_{s\to\infty} \mu(B_s(0)^c) = 0$. After repeating the same arguments presented in the example of the previous section, we may conclude that μ is not a ULP_∞-measure, a contradiction.

Conversely, let us assume that $\mu(B_\zeta(0)^c) = \infty$ for all $\zeta > 0$. Let $d_1, d_2 \in \mathbb{R}$ be with $\mu\text{-}\lim_{t\to\infty} f(t) = d_1$ and $\mu\text{-}\lim_{t\to\infty} f(t) = d_2$. From the definition of convergence in measure at infinity, for each $\epsilon, \delta > 0$ there exists $\zeta \geq 0$ such that

$$\mu(\{s \in B_\zeta(0)^c \mid \ |f(s) - d_1| > \epsilon \text{ or } |f(s) - d_2| > \epsilon\}) < \delta.$$

So,

$$\mu(Q = \{s \in B_\zeta(0)^c \mid \ |f(s) - d_1| \leq \epsilon \text{ and } |f(s) - d_2| \leq \epsilon\}) = \infty.$$

We may derive from the equality above that the set Q is not empty. After choosing $\epsilon < |d_1 - d_2|/2$ and taking any $s \in Q$ we see that $d_1 = d_2$. The verification of the cases where $d_1 = \infty$ or $d_2 = \infty$ are analogous. $\square$

Definition 4.1.3. Let μ be a ULP_∞-measure. The set $L_{0,c}(\mathscr{M})$ is defined as $L_{0,c}(\mathscr{M}) := \{f \in L_0(\mathscr{M}) \mid \mu\text{-}\lim_{t\to\infty} f(t) \text{ exists and is a real number}\}$.

We close this section with a result that can be obtained after revisiting the proof of Proposition 4.1.1.

Corollary 4.1.1. *Let μ be a ULP_∞-measure. Let $f \in L_{0,c}(\mathscr{M})$ and $\epsilon, \delta > 0$. Then, there exists $\zeta > 0$ such that $\mu(C_{\epsilon,f}(\zeta)) = \{s \in B_\zeta(0)^c \mid |f(s) - d| > \epsilon\}) < \delta$, and $\mu(F_{\epsilon,f}(\zeta) = \{s \in B_\zeta(0)^c \mid |f(s) - d| \leq \epsilon\}) = \infty$, where $\mu\text{-}\lim_{t\to\infty} f(t) = d$. In particular, there exists $\zeta > 0$ where the set $\{s \in B_\zeta(0)^c \mid |f(s) - d| \leq \epsilon\}$ is not empty.*

Remark 4.1.1. The result above explains from a different perspective why there is no uniqueness of the $\mu\text{-}\lim_{t\to\infty} f(t)$ for finite measures. The main problem is that when the measure μ is finite, the set $\{s \in B_\zeta(0)^c \mid |f(s) - d| \leq \epsilon\}$ may happen to be small (in the measure theory sense) or even empty (with ζ being large enough) even in the case when $\{s \in B_\zeta(0)^c \mid |f(s) - d| > \epsilon\}$ is also a small set for different values of d.

4.1.2 The main problem

For a fixed function $f \in L_1(\mathscr{M})$ with $|f(t)| > 0$ for all $t \in \mathscr{M}$ we consider the sets:

- $A_d(f) = \{g \in L_1(\mathscr{M}) \mid \mu\text{-}\lim_{t\to\infty} | \dfrac{g(t)}{f(t)} |= d\}$ where $0 \leq d \leq \infty$.
- $B(f) = \bigcup_{0<d<\infty} A_d(f)$.

The set $B(f)$ can be seen as the set of functions in $L_1(\mathscr{M})$ that are controlled at infinity (in measure) by the function f. Indeed, let $g \in B(f)$, and $0 < c < \infty$ such that $\mu\text{-}\lim_{t\to\infty} \mid g(t)/f(t) \mid = c$. By definition, for fixed $\epsilon, \delta > 0$ there exists $\zeta \geq 0$ such that $\mu(C_{\epsilon,g/f}(\zeta) = \{s \in B_\zeta(0)^c \mid \mid |(g/f)(s)| - c \mid > \epsilon\}) < \delta$. From the latter, we may derive the abovementioned control. More precisely,

$$(c - \epsilon)|f(t)| < |g(t)| < (c + \epsilon)|f(t)| \text{ for all } t \in B_\zeta(0)^c \text{ with } t \notin C_{\epsilon,g/f}(\zeta).$$

In the present chapter, we are interested in the investigation of the (maximal) lineability of the sets $A_0(f), A_\infty(f)$ and $B(f)$. However, at first we need to make sure that the sets are well-defined. In broad terms, this problem only makes sense once we may assure that the space $L_1(\mathscr{M})$ is infinite-dimensional and that it also contains a strictly positive function f. For this reason, we have to digress from these problems regarding lineability, and we need to focus temporarily our attention to the following two questions: (1) When does $L_1(\mathscr{M})$ contain a strictly positive function? (2) When is $L_1(\mathscr{M})$ infinite-dimensional? In the next two sections, we will proceed in order to answer the above questions for arbitrary measure spaces $(\mathscr{M}, \mathscr{A}, \mu)$.

4.2 The space $L_p(\mathscr{M}, \mathscr{A}, \mu)$

In this section $(\mathscr{M}, \mathscr{A}, \mu)$ is an arbitrary measure space. The measure μ defines an equivalence relation on $\mathscr{A}$ in the following way

$$A \sim B \iff \mu(A \setminus B) = \mu(B \setminus A) = 0.$$

As usual, under the light of the above relation, we write

$$\overline{A} = \{B \in \mathscr{A} \mid A \sim B\} \text{ and } \overline{\mathscr{A}} = \{\overline{A} \mid A \in \mathscr{A}\}.$$

Remark 4.2.1. If $A \sim B$, then $\mu(A) = \mu(A \cap B) + \mu(A \cap B^c) = \mu(A \cap B)$ (resp. $\mu(B) = \mu(B \cap A)$). Hence, $\mu(A) = \mu(B)$.

Now, let us recall a couple of definitions according to [19]. A measurable set $A \in \mathscr{A}$ is called an *atom* when satisfies the following two conditions:

- $\mu(A) > 0$.
- $F \subset A$ $(F \in \mathscr{A})$ and $\mu(F) < \mu(A) \Longrightarrow \mu(F) = 0$.

We shall say that μ is *purely atomic* (or simply *atomic*) if every measurable set of positive measure A of $\mathscr{A}$ contains an atom.

We close this section with the following:

Proposition 4.2.1. *Let $A, B \in \mathscr{A}$ with $A \sim B$. If A is an atom, then B is also an atom.*

Proof. Let $F \in \mathscr{A}$ where $F \subset B$ and $\mu(F) < \mu(B)$. Observe that

$$\mu(F \cap A^c) \leq \mu(B \cap A^c) = \mu(B \setminus A) = 0, \text{ since } A \sim B.$$

Then, $\mu(F) = \mu(F \cap A)$. On the other hand, by Remark 4.2.1 we have $\mu(A) = \mu(B)$. Thus,

$$\mu(F \cap A) = \mu(F) < \mu(B) = \mu(A).$$

Hence, $\mu(F) = \mu(F \cap A) = 0$, since A is an atom. The proof is now complete.

□

4.2.1 Necessary and sufficient conditions for $L_p(\mathscr{M}, \mathscr{A}, \mu)$ to be infinite-dimensional

In order to discuss lineability in $L_p(\mathscr{M})$ spaces $(1 \leq p < \infty)$ we do need to impose that the space $L_p(\mathscr{M})$ has infinite dimension. At first, we will establish necessary and sufficient conditions for the space $L_p(\mathscr{M})$ to be finite dimensional. At the end we will provide a formula to compute $\dim_{\mathbb{R}}(L_p(\mathscr{M}))$ in the case that $L_p(\mathscr{M})$ is infinite dimensional. We start with the following result:

Theorem 4.2.1. *The following assertions are equivalent.*

(i) $L_p(\mathscr{M})$ *is finite-dimensional.*

(ii) *The set* $\{\mu(E) \mid E \in \mathscr{A}\}$ *is finite.*

(iii) $\mathscr{M} = E_1 \uplus E_2 \uplus \cdots \uplus E_n$, *where each* E_i *is an atom.*

(iv) *The measure μ is atomic, and there exist measurable sets $E_1, \dots, E_n$ fulfilling the following condition: For every atom $A \in \mathscr{A}$ with finite measure, there exists $i \in \{1, \dots, n\}$ such that $A \sim E_i$.*

Corollary 4.2.1. *The space $L_p(\mathscr{M})$ is infinite-dimensional if and only if there exists a sequence $(E_n) \subset \mathscr{A}$ of pairwise disjoint sets such that $0 < \mu(E_n) < \infty$ for all $n \in \mathbb{N}$.*

Proof. The converse follows from the fact that the set $\{\chi_{E_n} \mid n \in \mathbb{N}\}$ is a linearly independent subset of $L_p(\mathscr{M})$. For the forward direction see Remark 4.2.2. □

The proof of the above theorem will rely on the following three technical lemmas.

Lemma 4.2.1. *Let us suppose that μ is a finite measure. If $\mathscr{M}$ is an atom, then $L_p(\mathscr{M}) = \operatorname{span}(\chi_{\mathscr{M}}) = \{\lambda \cdot \chi_{\mathscr{M}} \mid \lambda \in \mathbb{R}\}$.*

Proof. Let $f \in L_p(\mathscr{M})$. For each $\alpha \in \mathbb{R}$, we set $A_\alpha = \{t \in \mathscr{M} \mid f(t) \geq \alpha\}$ and $B_\alpha = \{t \in \mathscr{M} \mid f(t) \leq \alpha\}$. We define $S = \{\alpha \in \mathbb{R} \mid \mu(A_\alpha) = \mu(\mathscr{M})\}$ and $R = \{\alpha \in \mathbb{R} \mid \mu(B_\alpha) = \mu(\mathscr{M})\}$. For each $\alpha \in \mathbb{R}$, we have $A_\alpha \cup B_\alpha = \mathscr{M}$. If $\alpha \notin S \cup R$, then $\mu(A_\alpha), \mu(B_\alpha) < \mu(\mathscr{M})$. Hence, $\mu(A_\alpha) = \mu(B_\alpha) = 0$, since $\mathscr{M}$ is an atom. So, $\mu(M) = 0$, a contradiction. Thus, $\mathbb{R} = S \cup R$.

Now we claim that the sets S and R are nonempty. Let us suppose that $R = \varnothing$. Consequently, for each $n \in \mathbb{N}$, $\mu(B_n) = 0$. From the fact that $\mathscr{M} = \bigcup_{n=1}^\infty B_n$ and (B_n) is an increasing sequence of measurable sets, we have $\mu(\mathscr{M}) = \lim_{n\to\infty} \mu(B_n) = 0$ which is not possible. So, $R \neq \varnothing$. In the same fashion, one may check that $S \neq \varnothing$. Next, let $\alpha \in S$ and $\beta \in R$. If $\alpha > \beta$, then $\mathscr{M} = A_\alpha \uplus B_\beta \uplus \{t \in \mathscr{M} \mid \beta < f(t) < \alpha\}$. Thus, $\mu(\mathscr{M}) \geq \mu(A_\alpha) + \mu(B_\beta) = 2\mu(\mathscr{M})$ a contradiction. Hence, $\alpha \leq \beta$ for all $\alpha \in S$ and $\beta \in R$. So, $\sup S \leq \beta$ for all $\beta \in R$. Therefore, $\sup S \leq \inf R$. If there exists $\delta \in \mathbb{R}$ with $\sup S < \delta < \inf R$, then the condition $\mathbb{R} = S \cup R$ yields that either $\delta \in S$ or $\delta \in R$. In any case we achieve a contradiction. Set $\gamma = \sup S = \inf R$. For each $n \in \mathbb{N}$, one may find $\gamma_n \in R$ where $\gamma < \gamma_n < \gamma + 1/n$. Using that $B_\gamma = \bigcap_{n=1}^\infty B_{\gamma_n}$ and that (B_{γ_n}) is a decreasing sequence of measurable sets, we may conclude that $\mu(B_\gamma) = \lim_{n\to\infty} \mu(B_{\gamma_n}) = \mu(\mathscr{M})$. So, $\gamma \in R$. Similarly, one may verify that $\gamma \in S$.

Finally, we may notice that $\mu(\mathscr{M} \setminus A_\gamma) = \mu(\{t \in \mathscr{M} \mid f(t) < \gamma\}) = 0$, since $\mu(A_\gamma) = \mu(\mathscr{M})$ and $\mathscr{M} = A_\gamma \uplus (\mathscr{M} \setminus A_\gamma)$. Therefore, $\mu(\mathscr{M}) = \mu(B_\gamma) = \mu(B_\gamma \setminus (\mathscr{M} \setminus A_\gamma)) = \mu(\{t \in \mathscr{M} \mid f(t) = \gamma\})$. Hence, $f = \gamma \cdot \chi_{\mathscr{M}}$ a.e. Then $L_p(\mathscr{M})$ is a linear space with basis $\{\chi_{\mathscr{M}}\}$.

□

Lemma 4.2.2. *Let μ be an infinite measure. If $\mathscr{M}$ is an atom, then $L_p(\mathscr{M}) = \{0\}$.*

Proof. For each $E \in \mathscr{A}$, we have either $\mu(E) = 0$ or ∞. So, a step function $h = \sum_{i=1}^n \lambda_i \chi_{E_i}$ (with $\lambda_i \geq 0$) lies in $L_p(\mathscr{M})$ if and only if $\mu(E_i) = 0$ whenever $\lambda_i > 0$. More precisely,

$h \in L_p(\mathscr{M})$ if and only if $h = 0$ a.e. Since every non-negative measurable function f in $L_0(\mathscr{M})$ can be a approximated by a non-decreasing sequence of non-negative step functions, we may conclude that f itself lies in $L_p(\mathscr{M})$ if and only if $f = 0$ a.e. (we are also using the Monotone Convergence Theorem).

□

Lemma 4.2.3. *Let $E \in \mathscr{A}$. Then $\mathscr{A}_E = \{F \in \mathscr{A} \mid F \subset E\}$ is a σ-algebra in E and μ_E defined by $\mu_E(F) = \mu(F)$ for all $F \in A_E$ is a measure on $\mathscr{A}_E$. Furthermore, if $f \in L_0(\mathscr{M}, \mathscr{A}, \mu)$ (resp. $f \in L_p(\mathscr{M}, \mathscr{A}, \mu)$), then $f|_E \in L_0(E, \mathscr{A}_E, \mu_E)$ (resp. $f|_E \in L_p(E, \mathscr{A}_E, \mu_E)$).*

Proof. The details are left to the reader. □

Proof. (of Theorem 4.2.1)

$(iii) \Rightarrow (iv)$: Let $\mathscr{M} = E_1 \uplus E_2 \uplus \cdots \uplus E_n$ where each E_i is an atom. If $E \in A$ is such that $\mu(E) > 0$, then, the condition $\mu(E) = \sum_{j=1}^n \mu(E \cap E_j)$ implies that $\mu(E \cap E_i) > 0$ for some $i \in \{1, \ldots, n\}$. Besides that, since $(E \cap E_i) \subset E_i$ and E_i is an atom, we see that $\mu(E \cap E_i) = \mu(E_i)$. In particular $(E \cap E_i)$ is an atom in E. Hence, μ is an atomic measure. For the second part, let $E \in \mathscr{A}$ be an atom with finite measure. By the same argument presented earlier, we may infer that $\mu(E_i) = \mu(E \cap E_i) = \mu(E)$ for some $i \in \{1, \ldots, n\}$. Therefore, we have the equality $0 = (\mu(E) - \mu(E \cap E_i)) = \mu(E \setminus E_i) = \mu(E_i \setminus E)$, or equivalently, $E \sim E_i$.

$(iv) \Rightarrow (iii)$: Let us assume that μ is atomic, and let $E_1, \ldots, E_n$ be as in (iv). Due to Proposition 4.2.1 and Remark 4.2.1 we may assume without loss of generality that E_j is an atom of finite measure for each $j \in \{1, \ldots, n\}$, where $E_p \nsim E_q$ for $p, q \in \{1, \ldots, n\}$ with $p \neq q$. Now, we claim that that $\mu(E_p \cap E_q) = 0$ for distinct $p, q \in \{1, \ldots, n\}$. Indeed, if this is not the case, then $0 < \mu(E_p \cap E_q) = \mu(E_p) = \mu(E_q)$. So, $\mu(E_p \setminus E_q) = 0$, that is, $E_p \sim E_q$, a contradiction. Next, set

$$\tilde{E}_p = E_p \setminus (\bigcup_{\substack{j=1 \\ j \neq p}}^{n} (E_p \cap E_j)).$$

It is clear that $\mu(\tilde{E}_p) = \mu(E_p)$, otherwise, $\mu(E_p) = 0$. In particular, $\mu(\varnothing) = \mu(\tilde{E}_p \setminus E_p) = 0 = \mu(E_p) - \mu(\tilde{E}_p) = \mu(E_p \setminus \tilde{E}_p)$, that is, $\tilde{E}_p \sim E_p$. Hence, we may assume that all the E_j ($j \in \{1, \ldots, n\}$) are pairwise disjoint atoms of positive measure. Set $F = \mathscr{M} \setminus (\uplus_{p=1}^n E_p)$. If G is a measurable subset contained in F with $0 < \mu(G) < \infty$, then - since μ is atomic - we may assume without loss of generality that $G \sim E_i$ for some $i \in \{1, \ldots, n\}$. So, we arrive at the contradiction $0 = \mu(E_i \setminus G) = \mu(E_i) > 0$ (we have used that E_i is an atom and $(E_i \cap G) \subset (E_i \cap F) = \varnothing$). Thus, either $\mu(F) = 0$ or $\mu(F) = \infty$. If $\mu(F) = \infty$, then $\mathscr{M} = F \uplus (\uplus_{j=1}^n E_j)$ provides the desired decomposition. Otherwise, if $\mu(F) = 0$, we replace the atom E_1 with the atom $(E_1 \cup F)$, and we take $\mathscr{M} = (F \cup E_1) \uplus (\uplus_{j=1}^n E_j)$. The proof is now complete.

$(ii) \Rightarrow (iv)$: If μ is not an atomic measure, then there is a set $E \in \mathscr{A}$ $(\mu(E) > 0)$ where E does not contain atoms. Hence, there exists a measurable set E_1 contained in E with $0 < \mu(E_1) < \mu(E) \leq \infty$. Once again, since E_1 is not an atom, there exists a measurable set E_2 with $E_2 \subset E_1$ and $0 < \mu(E_2) < \mu(E_1)$. Inductively, we can construct a decreasing sequence (E_n) with $0 < \mu(E_{n+1}) < \mu(E_n)$ for all $n \in \mathbb{N}$. Thus, $\{\mu(E) \mid E \in \mathscr{A}\}$ is an infinite set, a contradiction. Hence, μ is atomic. Now, let us suppose that for each $n \in \mathbb{N}$ there is an atom E_n with finite measure, where $E_n \nsim E_m$ if $n \neq m$. As in the previous paragraph, we may assume without loss of generality that the atoms E_n are pairwise disjoint. By setting $F_n = \biguplus_{i=1}^{n} E_n$, we have $\mu(F_{n+1}) > \mu(F_n)$ for all $n \in \mathbb{N}$. The latter contradicts the hypothesis.

$(iii) \Rightarrow (ii)$: Let $\mathscr{M} = E_1 \uplus E_2 \uplus \cdots \uplus E_n$ be a partition of $\mathscr{M}$ in atoms. For a fixed $E \in \mathscr{A}$, we see that $\mu(E) = \mu(E \cap E_1) + \cdots + \mu(E \cap E_n)$. Since each term has only two possible outcomes, namely, $\mu(E \cap E_i) = 0$ or $\mu(E \cap E_i) = \mu(E_i)$, we may infer that the set $\{\mu(E) \mid E \in \mathscr{A}\}$ is finite.

$(iii) \Rightarrow (i)$: Let $\mathscr{M} = E_1 \uplus E_2 \uplus \cdots \uplus E_n$ be a partition of $\mathscr{M}$ in atoms. Fix $f \in L_p(\mathscr{M})$. By Lemma 4.2.3 we have $f|_{E_i} \in L_p(E_i, \mathscr{A}_{E_i}, \mu_{E_i})$ for each $i \in \{1, 2, \ldots, n\}$. On the one hand, if $\mu(E_i) < \infty$, then by Lemma 4.2.1, $f|_{E_i} = \gamma_i \chi_{E_i}$ for some $\gamma_i \in \mathbb{R}$. On the other hand, if $\mu(E_i) = \infty$, then $f|_{E_i} = 0$ by Lemma 4.2.2. Therefore, $f(t) = \gamma_1 \chi_{E_1}(t) + \cdots + \gamma_n \chi_{E_n}(t)$. In addition, $\operatorname{span}(\chi_{E_1}, \ldots, \chi_{E_n}) \supset L_p(\mathscr{M})$. Then, $L_p(\mathscr{M})$ is finite-dimensional.

$(i) \Rightarrow (iv)$: As we have seen in the proof of $(ii) \Rightarrow (iv)$, if μ is not an atomic measure then there exists a decreasing sequence of measurable sets (E_n) satisfying $0 < \mu(E_{n+1}) < \mu(E_n) < \infty$ for all $n \in \mathbb{N}$. Set $F_n = E_n \setminus E_{n+1}$. The sequence (F_n) is pairwise disjoint with $0 < \mu(F_n) < \infty$ for all $n \in \mathbb{N}$. Similarly, if $\mathscr{M}$ contains a sequence (G_n) of atoms with finite measure, we may assume (based on previous discussions) that this sequence is also pairwise disjoint. So, if (iv) does not hold, we may find a sequence (F_n) of pairwise disjoint sets such that $0 < \mu(F_n) < \infty$. Clearly, the set $\{\chi_{F_n} \mid n \in \mathbb{N}\}$ is a linearly independent subset of $L_p(\mathscr{M})$, a contradiction. $\square$

Remark 4.2.2. The forward direction in Corollary 4.2.1 follows from the fact that $L_p(\mathscr{M})$ being infinite-dimensional implies that (iv) is false, and consequently, as we have seen in the proof of $(i) \Rightarrow (iv)$, there exists the desired sequence (E_n).

We close this section with a result that will be very useful for our purposes.

Theorem 4.2.2. *Let us suppose that* $\dim_{\mathbb{R}}(L_p(\mathscr{M})) = \infty$. *Then* $\dim_{\mathbb{R}}(L_p(\mathscr{M})) = \operatorname{card}(S)^{\aleph_0}$ *where* $S = \{\overline{E} \in \overline{\mathscr{A}} \mid \mu(E) < \infty\}$.

Proof. First of all, note that the condition $\dim_{\mathbb{R}}(L_p(\mathscr{M})) = \infty$ implies, by Corollary 4.2.1, that $\operatorname{card}(S) \geq \aleph_0$. Hence, $\operatorname{card}(S)^{\aleph_0} \geq \mathfrak{c}$. Second, we may notice that $\chi_E = \chi_F$ a.e. if and only if $E \sim F$. Consequently, we may assure the existence of at least $\operatorname{card}(S)$ linearly independent functions on $L_p(\mathscr{M})$. Hence, $\operatorname{card}(S) \geq \dim_{\mathbb{R}}(L_p(\mathscr{M}))$. Now, we recall that, since $L_p(\mathscr{M})$ is a Banach space, then $\dim_{\mathbb{R}}(L_p(\mathscr{M})) \geq \mathfrak{c}$. Therefore, $\operatorname{card}(L_p(\mathscr{M})) = \dim_{\mathbb{R}}(L_p(\mathscr{M})) \geq \operatorname{card}(S)$.

On the other hand, it is known that in the case that X is an infinite-dimensional Banach space, we have the equality $\operatorname{card}(X) = d(X)^{\aleph_0}$, where $d(X)$ is the cardinality of the smallest dense subset of X (see [20, Lemma 2.8]). In particular, since $L_p(\mathscr{M})$ is also a Banach space, we have $\operatorname{card}(L_p(\mathscr{M})) = d(L_p(\mathscr{M}))^{\aleph_0}$. By cardinal arithmetic, and the estimate $d(L_p(\mathscr{M}))^{\aleph_0} = \operatorname{card}(L_p(\mathscr{M})) \geq \operatorname{card}(S)$, we get

$$\operatorname{card}(S)^{\aleph_0} \leq d(L_p(\mathscr{M}))^{\aleph_0 \times \aleph_0} = d(L_p(\mathscr{M}))^{\aleph_0} = \operatorname{card}(L_p(\mathscr{M})) = \dim_{\mathbb{R}}(L_p(\mathscr{M})).$$

Next, let us recall that $\overline{\operatorname{span}(D)} = L_p(\mathscr{M})$, where $D = \{\chi_E \mid \overline{E} \in S\}$. We claim that $\operatorname{card}(L_p(\mathscr{M})) \leq \operatorname{card}(S)^{\aleph_0}$. Indeed, since each element of $L_p(\mathscr{M})$ is the limit of a sequence in $\operatorname{span}(D)$, one may find an injective map from $L_p(\mathscr{M})$ into $[\operatorname{span}(D)]^{\aleph_0}$ = the set of all sequences of elements of $\operatorname{span}(D)$. At last, note that $\operatorname{card}(\operatorname{span}(D)) = \max\{\mathfrak{c}, \operatorname{card}(S)\}$. Thus, $\dim_{\mathbb{R}}(L_p(\mathscr{M})) = \operatorname{card}(L_p(\mathscr{M})) \leq \operatorname{card}([\operatorname{span}(D)]^{\aleph_0}) = \operatorname{card}(S)^{\aleph_0}$. The proof is now complete due to the above inequality. □

Remark 4.2.3. In the event that $\operatorname{card}(S) \leq \mathfrak{c}$, the theorem above allows us to conclude that either $\dim_{\mathbb{R}}(L_p(\mathscr{M})) = \mathfrak{c}$ or $\dim_{\mathbb{R}}(L_p(\mathscr{M})) < \infty$. In particular, since it is well known that the Borel σ-algebra in $\mathbb{R}$ has cardinality $\mathfrak{c}$, we may conclude that either $\dim_{\mathbb{R}}(L_p(\mathbb{R})) = \mathfrak{c}$ or $\dim_{\mathbb{R}}(L_p(\mathbb{R})) < \infty$ (here we consider $\mathbb{R}$ with the Borel σ-algebra) despite of the chosen measure. The same applies to any other measure space whose sigma-algebra has $\mathfrak{c}$ sets.

4.3 The existence of a strictly positive function on $L_p(\mathscr{M})$ for $1 \leq p < \infty$

Theorem 4.3.1. *Let $(\mathscr{M}, \mathscr{A}, \mu)$ be an arbitrary measure space. Then $L_p(\mathscr{M})$ contains strictly positive functions if and only if μ is σ-finite.*

Proof. Let us suppose that $f \in L_p(\mathscr{M})$ is strictly positive. Clearly, $\mathscr{M}$ can be written as a union of measurable sets in the following way: $\mathscr{M} = \bigcup_{n=1}^{\infty}\{t \in \mathscr{M} \mid |f(t)| > n^{-1/p}\}$. Since $f \in L_p(\mathscr{M})$, we have $\mu(\{t \in \mathscr{M} \mid |f(t)| > n^{-1/p}\}) < \infty$ for all $n \in \mathbb{N}$. Therefore, μ is σ-finite.

Conversely, let us suppose that $\mathscr{M} = \bigcup_{n\in\mathbb{N}} M_n$ with $\mu(M_n) = m_n < \infty$. Let us set $f = \sum_{n\in\mathbb{N}} \lambda_n \chi_{M_n}$, where $0 < (\lambda_n)^p m_n < 2^{-n}$ if $m_n > 0$, and $\lambda_n = 1$ when $m_m = 0$. By construction $f \in L_p(\mathscr{M})$ and it is also strictly positive. The proof is now complete. □

We finish this discussion on prerequisites for the problem to be well-posed with the following result which states that $L_p(\mathscr{M})$ is actually infinite-dimensional whenever μ is simultaneously σ-finite and a ULP_∞-measure.

Theorem 4.3.2. *Let $(\mathscr{M}, \mathscr{A}, \mu)$ be a measure space, where $\mathscr{M}$ is an unbounded subset of a real normed vector space $Y = (Y, +, \|\cdot\|)$, let $\mathscr{A}$ be the Borel σ-algebra in $\mathscr{M}$, and let μ be an arbitrary measure. Assume that μ is simultaneously σ-finite and a ULP_∞-measure. Then, $L_p(\mathscr{M})$ is infinite-dimensional. Moreover, $L_p(B_\zeta(0)^c)$ is infinite-dimensional for all $\zeta \geq 0$ - $B_\zeta(0)^c$ is the measure space induced by $(\mathscr{M}, \mathscr{A}, \mu)$.*

Proof. Let us suppose, by contradiction, that $L_p(\mathscr{M})$ is finite-dimensional. In this case, by Theorem 4.2.1, we have $\mathscr{M} = E_1 \cup \cdots \cup E_n$, where $E_1, \ldots, E_n$ are atoms. Since μ is σ-finite, we may conclude that $\mu(E_i) < \infty$ for each $i \in \{1, \ldots, n\}$. Hence, μ is a finite measure. The latter contradicts the characterization of ULP_∞-measures that was provided in Proposition 4.1.1. For the last part, we may apply the same arguments to the set $B_\zeta(0)^c$, since $\mu(B_\zeta(0)^c) = \infty$. □

4.4 The sets $A_0(f)$, $A_\infty(f)$ and $B(f)$

Henceforth, $(\mathscr{M}, \mathscr{A}, \mu)$ is a measure space, where $\mathscr{M} = (\mathscr{M}, +, \|\cdot\|)$ is a fixed unbounded subset of a real normed vector space $Y = (Y, +, \|\cdot\|)$, $\mathscr{A}$ is the Borel σ-algebra in $\mathscr{M}$ and μ is an arbitrary UPL_∞-measure which is also σ-finite. In addition, we fix a strictly positive function $f \in L_1(\mathscr{M})$.

4.4.1 B(f) is non-lineable

We remind the reader that $\mu(B_\zeta(0)^c) = \infty$ for all $\zeta > 0$.

Proposition 4.4.1. *Let $g : \mathscr{M} \to \mathbb{R}$ be a function. Let us suppose that $\mu\text{-}\lim_{t\to\infty} g(t) = d > 0$. Then, there exists $\zeta \geq 0$ such that the set $Z_g(\zeta) = \{s \in B_\zeta(0)^c \mid g(s) = 0\}$ has finite measure.*

Proof. Let $0 < \epsilon < d$ and $\delta > 0$. Since $\mu\text{-}\lim_{t\to\infty} g(t) = d > 0$, there is $\zeta \geq 0$ such that $C_{\epsilon,g}(\zeta) = \{s \in B_\zeta(0)^c \mid |g(s) - d| > \epsilon\}$ has measure less than δ. From the condition $0 < \epsilon < d$, we see that $Z_g(\zeta) \subset C_{\epsilon,g}(\zeta)$. Therefore, $Z_g(\zeta)$ has a finite measure. □

Corollary 4.4.1. *Let $g : \mathscr{M} \to \mathbb{R}$ be a function. Let us suppose that $\mu\text{-}\lim_{t\to\infty} |g(t)| = d > 0$. Then, for each $\zeta \geq 0$, either $P_g(\zeta)$ or $N_g(\zeta)$ has infinite measure, where $P_g(\zeta) = \{s \in B_\zeta(0)^c \mid g(s) > 0\}$ and $N_g(\zeta) = \{s \in B_\zeta(0)^c \mid g(s) < 0\}$. Moreover, if there exists $\zeta_0 \geq 0$ such that $\mu(P_g(\zeta)) < \infty$ for each $\zeta \geq \zeta_0$, then $\mu(N_g(\zeta)) = \infty$ for all $\zeta \geq \zeta_0$. In particular, $\mu(N_g(\zeta)) = \infty$ for all $\zeta \geq 0$.*

Proof. By the previous proposition, there exists $\zeta_1 \geq 0$ such that $\mu(Z_g(\zeta_1)) < \infty$. Let us suppose that for some $\zeta_2 \geq 0$ we have $\mu(P_g(\zeta_2)) < \infty$ and $\mu(N_g(\zeta_2)) < \infty$. Now, we set $\zeta = \max\{\zeta_1, \zeta_2\}$. Then, $\mu(B_\zeta(0)^c) = \mu(Z_g(\zeta)) + \mu(P_g(\zeta)) + \mu(N_g(\zeta)) < \infty$, since $Z_g(\zeta) \subset Z_g(\zeta_1)$, $P_g(\zeta) \subset P_g(\zeta_2)$ and $N_g(\zeta) \subset N_g(\zeta_2)$. The latter contradicts the hypothesis that μ is a ULP_∞ measure. $\square$

Corollary 4.4.2. *Let $g, h : \mathscr{M} \to \mathbb{R}$ be two functions. Suppose that $\mu\text{-}\lim_{t\to\infty} |g(t)| = d$ and $\mu\text{-}\lim_{t\to\infty} |h(t)| = c$, where $d, c > 0$. In addition, assume that $P_g(\zeta)$ has infinite measure for each $\zeta \geq 0$. Then, either $(P_g(\zeta) \cap P_h(\zeta))$ or $(P_g(\zeta) \cap N_h(\zeta))$ has infinite measure for each $\zeta \geq 0$.*

Proof. Let $\zeta_0 \geq 0$ be such that $\mu(Z_h(\zeta_0)) < \infty$. Now, for any $\zeta \geq \zeta_0$, the equality $\mu(P_g(\zeta)) = \mu(P_g(\zeta) \cap P_h(\zeta)) + \mu(P_g(\zeta) \cap N_h(\zeta)) + \mu(P_g(\zeta) \cap Z_h(\zeta))$ implies that either $(P_g(\zeta) \cap P_h(\zeta))$ or $(P_g(\zeta) \cap N_h(\zeta))$ has infinite measure. This completes the proof. $\square$

Remark 4.4.1. Under the light of Corollary 4.4.1, we may assume without loss of generality that $\mu(P_g(\zeta)) = \infty$ for all $\zeta \geq 0$. Thus, the relation $P_g(\zeta) = (P_h(\zeta) \cap P_g(\zeta)) \uplus (N_h(\zeta) \cap P_g(\zeta)) \uplus (Z_h(\zeta) \cap P_g(\zeta))$ tells us that either $\mu(P_h(\zeta) \cap P_g(\zeta)) = \infty$ or $\mu(N_h(\zeta) \cap P_g(\zeta)) = \infty$ for each $\zeta \geq 0$. Thus, by the same spirit of Corollary 4.4.1, we may assume that $\mu(P_h(\zeta) \cap P_g(\zeta)) = \infty$ for all $\zeta \geq 0$.

Proposition 4.4.2. *The set $B(f) \cup \{0\}$ only contains finite-dimensional subspaces of dimension 1. In particular, $B(f)$ is not lineable.*

Proof. Fix $g, h \in B(f)$. Now, let $0 < c, d < \infty$ be such that $\mu\text{-}\lim_{t\to\infty} \mid g(t)/f(t) \mid = c$ and $\mu\text{-}\lim_{t\to\infty} \mid h(t)/f(t) \mid = d$. We may assume without loss of generality that the $\mu(P(\zeta)) = \mu(P_{g/f}(\zeta) \cap P_{h/f}(\zeta)) = \infty$ for all $\zeta \geq 0$. We claim that $(dg - ch) \notin B(f)$. Indeed, suppose otherwise, that is, $(dg - ch) \in B(f)$. Consequently, there exists $b > 0$ such that $\mu\text{-}\lim_{t\to\infty} \mid (dg(t) - ch(t))/f(t) \mid = b$. Thus, for given $\epsilon, \delta > 0$, there is $\zeta_0 \geq 0$ such that the measure of the set $\{s \in B_\zeta(0)^c \mid |(dg(s) - ch(s))/f(s)| - b| > \epsilon\}$ is less than δ for all $\zeta \geq \zeta_0$.

Now, let us fix $0 < \epsilon < b/2$. There is $\zeta_1 \geq \zeta_0$ such that $\zeta \geq \zeta_1$ implies that

$$\mu(\{s \in B_\zeta(0)^c \mid |g(s)/f(s)| - c| > \epsilon/2d\}) < \delta \text{ and}$$

$$\mu(\{s \in B_\zeta(0)^c \mid |h(s)/f(s)| - d| > \epsilon/2c\}) < \delta.$$

Next, consider the set $R(\zeta) = F(\zeta) \cap P(\zeta)$, where

$$F(\zeta) = \{s \in B_\zeta(0)^c \mid |g(s)/f(s)| - c| \leq \epsilon/2d\} \cap \{s \in B_\zeta(0)^c \mid |h(s)/f(s)| - d| \leq \epsilon/2c\}.$$

Observe that

$$\mu(F(\zeta)^c) = \mu(\{s \in B_\zeta(0)^c \mid |g(s)/f(s)| - c| > \epsilon/2d\} \cup \{s \in B_\zeta(0)^c \mid |h(s)/f(s)| - d| > \epsilon/2c\}) < 2\delta.$$

Hence, $\mu(R(\zeta)) = \infty$, since $P(\zeta) = (P(\zeta) \cap F(\zeta)) \uplus (P(\zeta) \cap F(\zeta)^c)$.

Finally, if $s \in R(\zeta)$, we have

$$\left|\frac{dg(s) - ch(s)}{f(s)}\right| = \left|d\frac{g(s)}{f(s)} - c\frac{h(s)}{f(s)} - cd + cd\right| \leq d\left|\frac{g(s)}{f(s)} - c\right| + c\left|\frac{h(s)}{f(s)} - d\right| \leq \epsilon < b/2.$$

Therefore, $R(\zeta) \subset \{s \in B_\zeta(0)^c \mid |(dg(s) - ch(s))/f(s)| - b| > \epsilon\}$ for all $\zeta \geq \zeta_1$. This leads us to a contradiction. □

4.4.2 The set $A_0(f)$

Proposition 4.4.3. *The set $A_0(f)$ is $\mathfrak{c}$-lineable.*

Proof. For each $\alpha > 1$ we consider the function $g_\alpha(t) = f(t)/(1+\|t\|)^\alpha$. Since, $|g_\alpha(t)| \leq |f(t)|$ for all $t \in \mathscr{M}$, we may infer that $g_\alpha \in L_1(\mathscr{M})$ for all $\alpha > 1$. It is straightforward to verify that $g_\alpha \in A_0(f)$. The set $\{g_\alpha \mid \alpha > 1\}$ is linearly independent. Indeed, let $\lambda_1, \dots, \lambda_k$ be real numbers such that

$$\sum_{n=1}^{k} \lambda_n g_{\alpha_n} = 0.$$

The equality above yields

$$S(t) = \sum_{n=1}^{k} \lambda_n \frac{1}{(\|t\| + 1)^{\alpha_n}} = 0 \quad a.e \quad \text{on} \quad \mathscr{M}.$$

We may assume without loss of generality that $\alpha_i < \alpha_j$ if $i < j$. We set $P(t) = \prod_{n=1}^{k}(1+\|t\|)^{\alpha_n}$. From the relation $(P \cdot S) = 0$, we may infer after rearranging the terms from the highest to the smallest degree that

$$R(t) = \lambda_1 \prod_{n=2}^{k}(\|t\| + 1)^{\alpha_n} + \sum_{n=2}^{k} \lambda_n \prod_{\substack{i=1 \\ i \neq n}}^{k}(\|t\| + 1)^{\alpha_i} = 0 \quad a.e \quad \text{on} \quad \mathscr{M}.$$

Hence, if $\lambda_1 \neq 0$, then $\lim_{\|t\| \to \infty} |R(t)| = \infty$. In particular, for some $\zeta > 0$, $R(t) \neq 0$ for all $t \in B_\zeta(0)^c$. The condition $\mu(B_\zeta(0)^c) = \infty$ contradicts the fact that $R(t) = 0$ $a.e$ on $\mathscr{M}$. Recursively, we may conclude that $\lambda_n = 0$ for all $n = 1, \dots, k$. Clearly, $\text{span}(\{g_\alpha \mid \alpha > 1\}) \subset A_0(f)$. The proof is now complete. □

4.4.3 The case $d = \infty$

Proposition 4.4.4. *The set $A_\infty(f)$ is non-empty.*

Proof. For each $k \in \mathbb{N}$, let $n_k \in \mathbb{N}$ be such that

$$\int_{B_{n_k}(0)^c} |f| d\mu < 1/2^{2k}.$$

We may assume without loss of generality that $n_i < n_j$ if $i < j$. Let us set $g(t) = 2^k |f(t)|$ if $n_k \leq \|t\| < n_{k+1}$ and $g(t) = |f(t)|$ for $0 \leq \|t\| < n_1$. Thus, $\mu\text{-}\lim\limits_{t \to \infty} \mid g(t)/f(t) \mid = \infty$. Besides that,

$$g(t) = |f(t)| \chi_{B_{0,n_1}(0)} + \sum_{k=1}^{\infty} 2^k |f(t)| \chi_{B_{n_k, n_{k+1}}(0)}.$$

Hence,

$$\begin{aligned}\int_M |g| d\mu &= \int_{B_{0,n_1}(0)} |f| d\mu(t) + \sum_{k=1}^{\infty} \int_{B_{n_k,n_{k+1}}} 2^k |f| d\mu \\ &\leq \int_M |f| d\mu + \sum_{k=1}^{\infty} 2^k (1/2^{2k}) < \infty.\end{aligned}$$

Therefore, $g \in L_1(\mathscr{M})$. The proof is now complete. □

Our goal at this point it is to show that $A_\infty(f)$ is $\mathfrak{c}$-lineable. We start by providing a sufficient condition for a subset L (satisfying certain conditions) of $A_\infty(f)$ to be a basis of a linear subspace contained in $A_\infty(f) \cup \{0\}$.

Lemma 4.4.1. *Let L be a subset of $A_\infty(f)$ fulfilling the following condition:*

(i) If $g, h \in L$, then either $g \in A_\infty(h)$ or $h \in A_\infty(g)$.

(ii) Every function $g \in L$ is strictly positive.

Then L is a linearly independent subset of $A_\infty(f)$ and $\operatorname{span}(L) \subset A_\infty(f) \cup \{0\}$.

Proof. We start by claiming if $g \in A_\infty(h)$ and $h \in A_\infty(k)$, then $g \in A_\infty(k)$. Indeed, let us fix $N, \delta > 0$. Since $\mu\text{-}\lim_{t\to\infty} \mid g(t)/h(t) \mid = \infty$ and $\mu\text{-}\lim_{t\to\infty} \mid h(t)/k(t) \mid = \infty$, there is $\zeta > 0$ such that the sets $D_{\sqrt{N},|g/h|}(\zeta) = \{t \in B_\zeta(0)^c \mid |g(t)/h(t)| < \sqrt{N}\}$ and $D_{\sqrt{N},|h/k|}(\zeta) = \{t \in B_\zeta(0)^c \mid |h(t)/k(t)| < \sqrt{N}\}$ have measure less than $\delta/2$. Hence, the inclusion $\{t \in B_\zeta(0)^c \mid |g(t)/k(t)| < N\} \subset D_{\sqrt{N},|g/h|}(\zeta) \cup D_{\sqrt{N},|h/k|}(\zeta)$ yields $\mu(D_{N,|g/k|}(\zeta)) < \delta$ holds. Then, $\mu\text{-}\lim_{t\to\infty} \mid g(t)/k(t) \mid = \infty$.

Now, let us fix $g_1, g_2, \cdots, g_n \in L$ and non-zero scalars $\alpha_1, \cdots, \alpha_n \in \mathbb{R}$. Based on the first paragraph, we may assume without loss of generality that $g_i \in A_\infty(g_j)$ when $i < j$. The proof boils down to showing that $h = \alpha_1 g_1 + \cdots + \alpha_n g_n \in A_\infty(f)$. To this end, we first note that

$$\begin{aligned}\left|\frac{\alpha_1 g_1(t) + \cdots + \alpha_n g_n(t)}{f(t)}\right| &= \left|\frac{\alpha_1 g_1(t) + \cdots + \alpha_n g_n(t)}{g_1(t)}\right| \left|\frac{g_1(t)}{f(t)}\right| \\ &= \left|\alpha_1 + \alpha_2 \frac{g_2(t)}{g_1(t)} + \cdots + \alpha_n \frac{g_n(t)}{g_1(t)}\right| \left|\frac{g_1(t)}{f(t)}\right|.\end{aligned}$$

Next, let $\delta, N > 0$ be arbitrary. For each $i = 2, \cdots, n$, we set $J_i = \dfrac{|\alpha_1|}{2|\alpha_i|(n-1)}$. If the inequality $|g_i(t)/g_1(t)| < J_i$ holds for each $i = 2, \cdots, n$, then

$$\left|\alpha_2 \frac{g_2(t)}{g_1(t)} + \cdots + \alpha_n \frac{g_n(t)}{g_1(t)}\right| \leq |\alpha_2| \left|\frac{g_2(t)}{g_1(t)}\right| + \cdots + |\alpha_n| \left|\frac{g_n(t)}{g_1(t)}\right| <$$

$$|\alpha_2| J_2 + \cdots + |\alpha_n| J_n = (n-1)\frac{|\alpha_1|}{2(n-1)} = |\alpha_1|/2.$$

Hence, under these conditions, we obtain

$$\left|\alpha_1 + \alpha_2 \frac{g_2(t)}{g_1(t)} + \cdots + \alpha_n \frac{g_n(t)}{g_1(t)}\right| \geq |\alpha_1|/2.$$

In addition, if we assume that $|g_1(t)/f(t)| \geq 2N/|\alpha_1|$, we have

$$\left|\frac{\alpha_1 g_1(t) + \cdots + \alpha_n g_n(t)}{f(t)}\right| \geq N.$$

From the latter, by choosing $N_1 = 2N/|\alpha_1|$ and $N_i = 1/J_i$ for each $i = 2, \cdots, n$, it follows that

$$D_{N,|h/f|}(\zeta) \subset D_{N_1,|g_1/f|}(\zeta) \cup D_{N_2,|g_1/g_2|}(\zeta) \cup \cdots \cup \; D_{N_n,|g_1/g_n|}(\zeta).$$

The condition $\mu\text{-}\lim_{t\to\infty} \mid g_1(t)/g_i(t) \mid = \infty$ for each $i = 2, \cdots, n$ and $\mu\text{-}\lim_{t\to\infty} \mid g_1(t)/f(t) \mid = \infty$ guarantee the existence of an element $\zeta \geq 0$ such that

$$\mu(D_{N_1,|g_1/f|}(\zeta)) < \delta/n \text{ and } \mu(D_{N_i,|g_1/g_i|}(\zeta)) < \delta/n$$

for each $i = 2, \cdots, n$. Therefore, $\mu(D_{N,|h/f|}(\zeta)) < \delta$. Thus, by definition, $\mu\text{-}\lim_{t\to\infty} \mid h(t)/f(t) \mid = \infty$. Hence, $h \neq 0$. So, we may conclude that L is linearly independent and $\operatorname{span}(L) \subset A_\infty(f) \cup \{0\}$ as desired. □

Proposition 4.4.5. *The set $A_\infty(f)$ is $\mathfrak{c}$-lineable.*

Proof. Let α be in $(0, 2)$. For each $k \in \mathbb{N}$, let n_k as in the proof of Proposition 4.4.4. After repeating the same arguments presented in the proof of Proposition 4.4.4, we may conclude that the function $g_\alpha \in A_\infty(f)$, where $g_\alpha(t) = 2^{\alpha k}|f(t)|$ if $n_k \leq \|t\| < n_{k+1}$ and $g_\alpha(t) = |f(t)|$ when $0 \leq \|t\| < n_1$. In additon, if $\alpha, \beta \in (0, 2)$ and $\alpha > \beta$, then $\mu - \lim_{t\to\infty} \mid g_\alpha(t)/g_\beta(t) \mid = \infty$. Hence, if we set $L = \{g_\alpha \mid \; \alpha \in (0, 2)\}$, we see that L satisfies the conditions (i) and (ii) in Lemma 4.4.1. Therefore, $\operatorname{span}(L)$ is a subset of $A_\infty(f) \cup \{0\}$ and $\dim_{\mathbb{R}} \operatorname{span}(L) = \operatorname{card}(L) = \mathfrak{c}$. □

REFERENCES

1 V. I. Gurariy, "Subspaces and bases in spaces of continuous functions," *Dokl. Akad. Nauk*, vol. 167, pp. 595–598, 1966.

2 V. Fonf, V. Gurariy, and M. Kadets, "An infinite dimensional subspace of c[0,1] consisting of nowhere differentiable functions," *C. R. Acad. Bulgare Sci.*, vol. 52, pp. 13–16, 1999.

3 G. Araújo, L. Bernal-González, G. A. Muñoz-Fernández, J. A. Prado-Bassas, and J. Seoane-Sepúlveda, "Lineability in sequence and function spaces," *Studia Math.*, vol. 237, no. 2, pp. 119–136, 2017.

4 G. Botelho, V. Fávaro, D. Pellegrino, and J. Seoane-Sepúlveda, "$L_p[0,1] \setminus \cup_{q>p} L_q[0,1]$ is spaceable for every $p > 0$," *Linear Algebra Appl.*, vol. 436, p. 2963–2965, 2012.

5 D. Cariello and J. B. Seoane-Sepúlveda, "Basic sequences and spaceability in l_p spaces," *J. Funct. Anal.*, vol. 266, no. 6, pp. 3797–3814, 2014.

6 V. P. Fonf and C. Zanco, "Almost overcomplete and almost overtotal sequences in Banach spaces," *J. Math. Anal. Appl.*, vol. 420, no. 1, pp. 94–101, 2014.

7 A. Aizpuru, C. Pérez-Eslava, and J. Seoane-Sepúlveda, "Linear structure of sets of divergent sequences and series," *Linear Algebra Appl.*, vol. 418, no. 2, pp. 595–598, 2006.

8 P. Leonetti, T. Russo, and J. Somaglia, "Dense lineability and spaceability in certain subsets of ℓ_∞," *Bull. Lond. Math. Soc.*, vol. 55, pp. 2283–2303, 2023.

9 G. Botelho, D. Diniz, V. Fávaro, and D. Pellegrino, "Spaceability in Banach and quasi-Banach sequence spaces," *Linear Algebra Appl.*, vol. 434, no. 5, pp. 1255–1260, 2011.

10 R. Aron, L. Bernal-González, D. Pellegrino, and J. Seoane-Sepúlveda, *Lineability: The Search for Linearity in Mathematics.* Monographs and Research Notes in Mathematics, Chapman and Hall/CRC, New York, 2015.

11 G. Araújo and A. Barbosa, "A general lineability criterion for complements of vector spaces," *Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Mat.*, vol. 118, no. 1, p. 5, 2023.

12 D. Pellegrino and A. Raposo, "Pointwise lineability in sequence spaces," *Indag. Math.*, vol. 32, no. 2, pp. 536–546, 2021.

13 V. Fávaro, D. Pellegrino, and D. Tomaz, "Lineability and spaceability: a new approach," *Bull. Braz. Math. Soc.*, vol. 51, pp. 27–46, 2020.

14 E. Kreyzig, *Introductory Functional Analysis with Applications.* Wiley classics library, Wiley India Pvt. Limited, 2007.

15 N. Jacobson, *Lectures in Abstract Algebra: Linear algebra.* Lectures in Abstract Algebra, Van Nostrand, 1951.

16 V. I. Gurariy and L. Quarta, "On lineability of sets of continuous functions," *J. Math. Anal. Appl.*, vol. 294, no. 1, pp. 62–72, 2004.

17 L. Bernal-González, H. Cabana-Méndez, G. Muñoz-Fernández, and J. Seoane-Sepúlveda, "On the dimension of subspaces of continuous functions attaining their maximum finitely many times," *Trans. Amer. Math. Soc.*, vol. 373, pp. 3063–3083, 2020.

18 R. M. Aron and P. Hájek, "Odd degree polynomials on real Banach spaces," *Positivity*, vol. 11, no. 1, pp. 143–153, 2007.

19 R. A. Johnson, "Atomic and nonatomic measures," *Proc. Amer. Math. Soc.*, vol. 25, no. 3, pp. 650–655, 1970.

20 T. Bartoszyński, M. Džamonja, L. Halbeisen, E. Murtinová, and A. Plichko, "On bases in Banach spaces," *Studia Math.*, vol. 170, no. 2, pp. 147–171, 2005.